NOTES ON RADIAL AND QUASIRADIAL FOURIER MULTIPLIERS

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These notes were prepared for lectures at the Summer School on Nonlinear Analysis, Function Spaces and Applications (NAFSA 10), June 9-15, 2014 in Třešt', Czech Republic. They are based on material in various joint papers ([1], [18], [23], [24], [29], [31], [32]). The notes could not have been written without the extensive contributions of my coauthors Jong-Guk Bak, Gustavo Garrigós, Yaryong Heo, Sanghyuk Lee, Fedya Nazarov, and Keith Rogers. Of course any deficiencies and errors in these notes are my responsibility.

1. Fourier multipliers

Given a translation invariant operator T which is bounded from $\mathcal{S}(\mathbb{R}^d)$ to some $L^q(\mathbb{R}^d)$ there is a tempered distribution K so that $Tf = K * f$ for all $f \in \mathcal{S}(\mathbb{R}^d)$ (see [48]). We denote by Conv_p^q the space of all $K \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|K * f\|_q \le C\|f\|_p$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$, and set

$$\|K\|_{\mathrm{Conv}_p^q} = \sup\{\|K * f\|_q : f \in \mathcal{S}, \|f\|_p = 1\}$$

The convolution operators can also be written as a multiplier transformation

$$K * f = \mathcal{F}^{-1}[m\widehat{f}]$$

where the Fourier transform of a Schwartz function in $\mathbb{R}^d$ is given by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int f(y)e^{-i\langle y,\xi\rangle}dy$$

and $\mathcal{F}^{-1}[g](x) = (2\pi)^{-d}\int g(\xi)e^{i\langle x,\xi\rangle}d\xi$ is the inverse Fourier transform of g. The space of Fourier transforms of distributions in Conv_p^q is denoted by M_p^q and the Fourier transform is an isometric isomorphism $\mathcal{F} : \mathrm{Conv}_p^q \to M_p^q$, i.e. $\|m\|_{M_p^q} = \|\mathcal{F}^{-1}[m]\|_{\mathrm{Conv}_p^q}$. We also abbreviate

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$\mathrm{Conv}_p = \mathrm{Conv}_p^p$ and $M_p = M_p^p$. Effective characterizations of the spaces Conv_p^q or M_p^q are known only in a few cases, see [25], [48]:

- Conv_1^1 is the space of finite Borel measures (with the total variation norm).
- For $1 < p \le \infty$, $\mathrm{Conv}_1^q = L^q$, with identifications of the norms.
- $M_2 = L^\infty$, with identifications of the norms.

Moreover, $M_p^q = \{0\}$ for $p > q$, and $M_p^q = M_{q'}^{p'}$, $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$.

Of particular interest are the *radial* Fourier multipliers on $\mathbb{R}^d$

$$m(\xi) = h(|\xi|)\,.$$

We shall assume throughout that $d \ge 2$. The radial multipliers correspond to operators commuting with translations and rotations, i.e. convolution operators with radial kernels, and also to spectral multiplier operators for the Laplace operator, via

$$h(\sqrt{-\Delta})f = \mathcal{F}^{-1}[h(|\cdot|)\widehat{f}]\,.$$

A great deal of research has been done on a class of radial model multipliers which are now called of Bochner-Riesz type. Given a smooth bump function ϕ supported in $(-1/2, 1/2)$ we define

$$h_\delta(s) = \phi(\tfrac{1-s}{\delta})$$

so that $h_\delta(|\cdot|)$ is supported on an annulus of width $\sim \delta$.

Conjecture I: For $1 < p < \frac{2d}{d+1}$, one has

$$\|h_\delta(|\cdot|)\|_{M_p} \lesssim \delta^{-\lambda(p)}, \quad \lambda(p) = d(1/p - 1/2) - 1/2 \tag{1.1}$$

This is referred to as the Bochner-Riesz conjecture since a resolution would prove that the Bochner-Riesz means of the Fourier integral

$$R^\lambda f(x) = \frac{1}{(2\pi)^d}\int_{|\xi|\le t}\big(1 - \frac{|\xi|}{t}\big)^\lambda \widehat{f}(\xi) e^{i\langle x,\xi\rangle} d\xi\,.$$

converge to f in $L^p(\mathbb{R}^d)$ if $\lambda > \lambda(p)$. Inequality (1.1) is known in two dimensions for the full range, and various partial results have been proved in higher dimensions (see [15],[12], [17], [5], [6], [55], [28], [7]). For the relationships of various related conjectures see [51], [52].

A somewhat more general problem involves classical function spaces measuring regularity, in particular the L^2 type Sobolev spaces $\mathcal{L}^2_\alpha$ or the Besov spaces $B^2_{\alpha,p}$. The following estimate would be optimal within this class of spaces.

Conjecture II: Let h be supported in a compact subinterval J of $(0,\infty)$. The estimate

$$\|h(|\cdot|)\|_{M_p} \lesssim \|h\|_{B^2_{\alpha,p}}, \quad \alpha = d(1/p - 1/2) \tag{1.2}$$

holds for $1 \le p < \frac{2d}{d+1}$.

Partial results with the condition $\alpha > d(1/p - 1/2)$ and related square-function estimates are in [9], [11], [13], [29], and a proof of the endpoint $B^2_{d(1/p-1/2),p}$ bound, in the range $1 < p < \frac{2(d+1)}{d+3}$ can be found in [30].

Inequality (1.2) is still far from a characterization of radial multipliers (even those supported in $(1/2, 2)$). Consider the multiplier $h_t(s) = \chi(s)e^{-its}$ where $\chi \in C^\infty_c(1/2, 2)$. It is closely related to the solution operator for the wave equation. It is known (see [40]) that $\|h_t(|\cdot|)\|_{M_p} \lesssim t^{(d-1)(1/p-1/2)}$ for $t > 1$, $1 \le p \le 2$, but $\|h_t\|_{B^2_{\alpha,q}} \gtrsim t^\alpha$ and thus the optimal M_p bound cannot be derived from (1.2). An affirmative answer to the following conjecture would close this gap.

Conjecture III: Let $1 < p < \frac{2d}{d+1}$, and let h be supported in a compact subinterval J of $(0,\infty)$. Then it

$$\|h(|\cdot|)\|_{M_p} \approx \big\|\mathcal{F}^{-1}[h(|\cdot|)]\big\|_p \tag{1.3}$$

(with the implicit constants depending on J)?

Given that the weaker conjectures I and II are not completely resolved, we may have, for Conjecture III, to settle for smaller ranges of p. It has been shown to hold in [23] for the range $d \ge 4$, $p < \frac{2(d-1)}{d+1}$, and no such result is currently known in dimensions 2 and 3. As in [23] the proof of this conjecture would likely lead to a simple characterization of all radial $\mathcal{F}L^p$ multipliers for $p < \frac{2d}{d+1}$, as in:

Conjecture IV: Let $m = h(|\cdot|)$ be a bounded radial function on $\mathbb{R}^d$ and define the convolution operator T_h on $\mathbb{R}^d$ by

$$\widehat{T_h f}(\xi) = h(|\xi|)\widehat{f}(\xi).$$

Let $1 < p < \frac{2d}{d+1}$. Let η be any nontrivial Schwartz function on $\mathbb{R}^d$ and let ϕ be any nonzero C^∞_c function compactly supported on $(0,\infty)$. Then T_h is bounded on L^p if and only if

$$\sup_{t>0} t^{d/p}\big\|T_h[\eta(t\cdot)]\big\|_p < \infty$$

The resolution of this conjecture would imply that in the above range T_h is bounded on L^p if and only if T_h is bounded on the subspace L^p_{rad} of radial L^p functions. By Fefferman's theorem [16] on the ball multiplier (or some variant of it) this is clearly false for $\frac{2d}{d+1} \le p \le 2$. Similar arguments show that the condition $p < \frac{2d}{d+1}$ is necessary in the three previous conjectures.

These notes. In §2 we consider the case of general quasiradial multipliers in M^2_p and discuss a characterization for the case when the under a decay assumption on the Fourier transform of the surface measure of Σ_ρ. When ρ is the Minkowski functional of a set with smooth boundary (then $P = I$) we examine further the M^q_1 classes and express the condition $\mathcal{F}^{-1}[m] \in L^q$ in terms of the one dimensional Fourier transform of h. This is done in §3. In §4 we discuss a sharp theorem from [18] on the L^p boundedness of convolution operators with radial kernels when acting on radial functions, and in §5 we prove the same result for radial multipliers acting on general functions. We shall focus on the case where the multipliers are supported in $(1/2, 2)$.

2. L^2 Fourier restriction theorems and quasiradial M^2_p multipliers

In this section we shall first discuss a general version of the Stein-Tomas restriction theorem and then deduce as a consequence a straightforward characterization of radial and quasi radial multipliers in M^2_p, for the p-range of the L^2 restriction theorem.

2.1. *L^2 Fourier restriction theorems.* In the 1960's Stein observed that the Fourier transform of an L^p function can, for a nontrivial range of $p > 1$ be restricted to some compact hypersurfaces *with suitable curvature assumptions.* An almost sharp theorem for the sphere was proved by Tomas [56], with an endpoint version due to Stein [46]. Further results are in Greenleaf [20] and many other papers. Here we present the rather general setup by Mockenhaupt [38] and by Mitsis [37], with the endpoint version in a joint paper with Bak [1].

We are given a compactly supported Borel measure μ satisfying the Fourier decay bound

$$|\widehat{\mu}(\xi)| \lesssim (1 + |\xi|)^{-\beta/2} \tag{2.1}$$

and the α-upper-regularity condition

$$|\mu(B(x,r))| \lesssim r^\alpha \tag{2.2}$$

for all balls $B(x,r)$ with radius r.

Remark. For measures satisfying (2.2) the Hausdorff dimension of the support is at least α. For measures satisfying (2.1) the Hausdorff dimension of the support is at least β. The latter holds since for $\gamma < \beta$ the γ-energy

$$I_\gamma(\mu) = \iint |x-y|^{-\gamma} d\mu(x) d\mu(y) = C \int |\xi|^{\gamma-d} |\widehat{\mu}(\xi)|^2 d\xi$$

is finite and for the Hausdorff dimension of $\mathrm{supp}(\mu)$ we have

$$\dim(\mathrm{supp}(\mu)) = \sup\{\gamma : I^\gamma(\mu) < \infty\}$$

(see [35], [58]). For this reason the number

$$\dim_{\mathcal{F}}(\mu) = \sup\{\beta : \sup_\xi (1+|\xi|)^{\beta/2} |\widehat{\mu}(\xi)| < \infty\}$$

is often called the *Fourier dimension* of μ.

Theorem 2.1. *Let μ be as in* (2.1), (2.2). *Then*

$$\int |\widehat{f}(\xi)|^2 d\mu \lesssim \|f\|_p^2, \quad p \le p_{\mathrm{cr}} = \frac{4(d-\alpha)+2\beta}{4(d-\alpha)+\beta}\,. \tag{2.3}$$

The L^p norm on the right can be replaced by the smaller $L^{p,2}$ Lorentz-norm.

Proof. Let T denote the Fourier transform, as an operator mapping L^p to $L^2(d\mu)$ (and as such a priori defined for L^1 functions). The proofs of Mockenhaupt and Mitsis, for the open range $(1, p_{\mathrm{cr}})$, follow Tomas' argument in [56]. One first observes that

$$T^*Tf = f * [\widehat{\mu}(-\cdot)]\,. \tag{2.4}$$

Indeed,

$$\begin{aligned} \langle T^*Tf, g\rangle &= \int \widehat{f}(\xi)\overline{\widehat{g}(\xi)} d\mu(\xi) \\ &= \int \widehat{f}(\xi)\widehat{\overline{g}}(-\xi) d\mu(\xi) \\ &= \int \overline{g}(y) \int \widehat{f}(\xi) e^{i\langle \xi, y\rangle} d\mu(\xi)\, dy \\ &= (2\pi)^d \langle \mathcal{F}^{-1}[\widehat{f}\, d\mu], g\rangle \;=\; f * [\widehat{\mu}(-\cdot)]\,. \end{aligned}$$

Thus

$$\mathcal{F} : L^{p,q} \to L^2(d\mu) \iff \widehat{\mu}* : L^{p,q} \to L^{p',q'} . \tag{2.5}$$

Following Tomas split $\widehat{\mu} = \sum_{j=0}^{\infty} \widehat{\mu}_j$, where $\widehat{\mu}_0$ is supported in $\{|\xi| \le 2\}$ and, for $j \ge 1$

$$\operatorname{supp}(\widehat{\mu}_j) \subset \{2^{j-1} \le |\xi| \le 2^{j+1}\}.$$

This decomposition can be arranged such that $\mu_0 = \mu * \phi_0$, and, for $j \ge 1$, $\mu_j = \mu * 2^{jd}\phi(2^j(x))$, for suitable $\phi_0, \phi \in \mathcal{S}(\mathbb{R}^d)$.

It is easy to see that $\|f * \mu_0\|_{p'} \lesssim \|f\|_p$ for $1 \le p \le 2$. By (2.1),

$$\|\widehat{\mu}_j\|_\infty \lesssim 2^{-j\beta/2}$$

for $j \ge 1$, and hence

$$\|f * \widehat{\mu}_j\|_\infty \lesssim 2^{-j\beta/2} \|f\|_1 . \tag{2.6}$$

By (2.2) we get, with $N \gg d$,

$$\begin{aligned}
|\mu_j(x)| &\lesssim \int \frac{2^{jd}}{(1 + 2^j|x-y|)^N} d\mu(y) \\
&\lesssim \sum_{l=0}^{j} \mu(B(x, 2^{-j+l})2^{jd}2^{-lN} + C_N 2^{j(d-N)} \\
&\lesssim \sum_{l=0}^{j} 2^{j(d-\alpha)} 2^{-l(N-\alpha)} + C_N 2^{j(d-N)} \lesssim 2^{j(d-\alpha)}
\end{aligned}$$

and hence

$$\|f * \widehat{\mu}_j\|_2 \lesssim \|\mu_j\|_\infty \|f\|_2 \lesssim 2^{j(d-\alpha)} \|f\|_2 . \tag{2.7}$$

Interpolating (2.6) and (2.7) we get

$$\|f * \widehat{\mu}_j\|_{p'} \lesssim 2^{-j(2(d-\alpha)+\frac{\beta}{2} - \frac{2(d-\alpha)+\beta}{p})} \|f\|_p$$

where the exponent is negative when $p < p_{\mathrm{cr}}$. Sum in j to get the asserted inequality (2.3) for $p < p_{\mathrm{cr}}$.

For the endpoint inequality the familiar analytic families interpolation argument (see e.g. [46] does not seem to work in the generality considered here, moreover it does not seem to yield the better Lorentz space bound. Instead one uses real methods (see [26], [21] for related arguments).

A familiar argument by Bourgain [4] yields the restricted weak type estimate

$$\big\|f * \widehat{\mu}\big\|_{L^{p'_{\mathrm{cr}},\infty}} \lesssim \|f\|_{L^{p_{\mathrm{cr}},1}} \tag{2.8}$$

or equivalently, for any f with $|f| \le \chi_E$

$$\operatorname{meas}(\{x : |f * \widehat{\mu}| > \lambda\}) \lesssim \Big(\frac{|E|^{1/p_{\rm cr}}}{\lambda}\Big)^{p'_{\rm cr}}.$$

To see this observe that for any $R > 1$

$$\Big|\sum_{2^j > R} f * \widehat{\mu}_j(x)\Big| \le C_0 \sum_{2^j > R} 2^{-j\beta/2} \|f\|_1 \le C_0' R^{-\beta/2}|E|.$$

Choose $R = R_\lambda$ so that $C_0' R_\lambda^{-\beta/2}|E| = \lambda/2$ (and thus $R_\lambda \approx (|E|/\lambda)^{2/\beta}$). Then

$$\begin{aligned}
&\operatorname{meas}\big(\{x : |f * \widehat{\mu}| > \lambda\}\big)\\
&\lesssim \operatorname{meas}\big(\{x : |\sum_{2^j \le R_\lambda} f * \widehat{\mu}_j(x)| > \lambda/2\}\big)\\
&\lesssim \lambda^{-2}\Big\|\sum_{2^j \le R_\lambda} f * \widehat{\mu}_j\Big\|_2^2 \lesssim \lambda^{-2}\big[R_\lambda^{(d-a)}\|f\|_2\big]^2\\
&\lesssim \lambda^{-2}(|E|/\lambda)^{4(d-a)/\beta}|E| \lesssim |E|^{\frac{4(d-a)+\beta}{\beta}}\lambda^{-\frac{4(d-a)+2\beta}{\beta}}
\end{aligned}$$

which is $(|E|^{1/p_{\rm cr}}/\lambda)^{p'_{\rm cr}}$.

The restricted weak type estimate implies, by (2.5),

$$\mathcal{F} : L^{p_{\rm cr},1} \to L^2(d\mu)\,. \tag{2.9}$$

To upgrade this result to a strong type $L^{p_{\rm cr}} \to L^2(d\mu)$ (or even $L^{p_{\rm cr},2} \to L^2(d\mu)$) bound we first prove an $L^{p_{\rm cr},1} \to L^2$ estimates for the convolutions $f * \widehat{\mu}_j$. We claim that (2.9) implies

$$\|f * \widehat{\mu}_j\|_2 \lesssim 2^{j(d-\alpha)/2}\|f\|_{L^{p_{\rm cr},1}}\,. \tag{2.10}$$

Indeed, since $\|\mu_j\|_\infty = O(2^{j(d-\alpha)})$,

$$\|f * \widehat{\mu}_j\|_2 \lesssim \|\widehat{f}\,\mu_j\|_2 \lesssim 2^{j(d-\alpha)/2}\Big(\int |\widehat{f}(\xi)|^2|\mu_j|d\xi\Big)^{1/2}$$

and, from (2.9),

$$\begin{aligned}
&\Big(\int |\widehat{f}(\xi)|^2|\mu_j|d\xi\Big)^{1/2}\\
&\lesssim \Big(\int \frac{2^{jd}}{(1+2^j|\xi|)^N}\int |\widehat{f}(\eta+\xi)|^2 d\mu(\eta)d\xi\Big)^{1/2} \lesssim \|f\|_{L^{p_{\rm cr},1}}.
\end{aligned}$$

Thus (2.10) follows. By duality we also get the $L^2 \to L^{p'_{\rm cr},\infty}$ estimate, and then, by the Marcinkiewicz interpolation theorem,

$$\|f * \widehat{\mu}_j\|_{\tilde{q}} \lesssim 2^{j(d-\alpha)/2}\|f\|_{\tilde{p}} \tag{2.11}$$

for $(1/\tilde{p}, 1/\tilde{q})$ on the open line segment joining the points $(1/p_{\mathrm{cr}}, 1/2)$ and $(1/2, 1/p'_{\mathrm{cr}})$.

We now repeat Bourgain's argument above, interpolating the bounds (2.11) with the estimate (2.6). This gives the bound

$$\big\|f * \widehat{\mu}\big\|_{L^{q,\infty}} \lesssim \|f\|_{L^{p,1}} \tag{2.12}$$

for $(1/p, 1/q)$ on an open line segment which is parallel to the diagonal $\{x, x\}$, and has midpoint $(1/p_{\mathrm{cr}}, 1/p'_{\mathrm{cr}})$.

Using the general Marcinkiewicz interpolation theorem again (on this line segment) we get

$$\big\|f * \widehat{\mu}\big\|_{L^{p'_{\mathrm{cr}},r}} \lesssim \|f\|_{L^{p_{\mathrm{cr}},r}} \tag{2.13}$$

for all r. Applying this for $r = 2$, and then applying (2.5) one more time yields

$$\mathcal{F} : L^{p_{\mathrm{cr}},2} \to L^2(d\mu)\,.$$

$\square$

2.2. *Quasiradial multipliers in M^2_p.* Some theorems for radial multipliers can be generalized for *quasiradial* multipliers, which do not possess any group invariance properties. They are given by

$$m(\xi) = h(\rho(\xi))$$

where ρ is a suitable *P-homogeneous distance function.*

To define this notion let P be a real $d \times d$ matrix whose eigenvalues have positive real part and let $t^P = \exp(P \log t)$. ρ is a P-homogeneous distance function if

$$\rho(t^P \xi) = t\rho(\xi)$$

for all $t > 0$, and ρ is continuous in $\mathbb{R}^d$ and $\rho(\xi) > 0$ for $\xi \neq 0$. In addition we shall throughout assume that ρ belongs to $C^\infty(\mathbb{R}^d \setminus \{0\}$.

Let

$$\Sigma_\rho = \{\xi : \rho(\xi) = 1\} \tag{2.14}$$

and let σ be the surface measure on Σ_ρ. It will be convenient to use generalized polar coordinates $\xi = s^P \xi'$ with $\rho(\xi) = s$, $\xi' \in \Sigma_\rho$; then

$$\int g(x)dx = \int_0^\infty \int_{\Sigma_\rho} g(s\xi') \frac{d\sigma(\xi')}{\langle P\xi', n(\xi')\rangle} s^{\nu-1} ds$$

where $n(\xi')$ is the outer normal vector at $\xi' \in \Sigma_\rho$ and

$$\nu = \mathrm{trace}(P).$$

Euler's homogeneity relation in this setting becomes $\rho(\xi) = \langle P\xi, \nabla\rho(\xi)\rangle$ so that $\langle P\xi', n(\xi')\rangle$ is bounded above and below on Σ_ρ.

We now discuss a simple characterization of quasiradial M^2_p multipliers in the case that the Fourier dimension of Σ_ρ is positive. Let σ be surface measure on Σ_ρ. We assume

$$\sup_{\xi\in\mathbb{R}^d} |\xi|^{\beta/2}|\widehat{\sigma}(\xi)| < \infty \tag{2.15}$$

for some $\beta > 0$. As Σ_ρ is a hypersurface we can apply Theorem 2.1 with $\alpha = d-1$ and get

$$\Big(\int_{\Sigma_\rho} |\widehat{g}(\xi')|^2 d\sigma(\xi')\Big)^{1/2} \lesssim \|f\|_{L^p(\mathbb{R}^d)}, \quad 1 \le p \le \frac{2\beta+4}{\beta+4} \tag{2.16}$$

(*cf.* [20]). For example, (2.15) holds with $\beta/2 = (d-1)/m$ if Σ_ρ is convex and all tangent lines to Σ_ρ have contact order of at least m with Σ_ρ (see e.g. [8]). One version of the following observation was stated in [18] although it had been known as a "folk result" for quite some time.

Theorem 2.2. *Suppose that* (2.15) *holds and* $1 < p \le \frac{2\beta+4}{\beta+4}$ *. Then the operator*

$$f \mapsto \mathcal{F}^{-1}[h(|\cdot|)\widehat{f}]$$

extends to a bounded operator from $L^p(\mathbb{R}^d)$ *to* $L^2(\mathbb{R}^d)$ *if and only if*

$$\sup_{t>0} t^{\nu(\frac1p-\frac12)}\Big(\int_t^{2t} |h(\rho)|^2 \frac{d\rho}{\rho}\Big)^{1/2} < \infty. \tag{2.17}$$

Proof. If h is supported in $(1,4)$ then $h\circ\rho \in M^2_p$ immediately implies that $h\circ\rho \in L^2(\mathbb{R}^d)$, by testing the operator on suitable Schwartz functions. Thus, by polar coordinates, $h \in L^2$. By scaling, $\|m(t^P\cdot)\|_{M^2_p} = t^{\nu(1/p-1/2)}\|m\|_{M^2_p}$ and the necessity of the condition (2.17) follows.

For the sufficiency assume first that h_t is supported in $[t, 4t]$. Then using polar coordinates and the restriction theorem (2.16) wee see that

$$\begin{aligned}
\|h_t\circ\rho\,\widehat{f}\|_2 &\lesssim \Big(\int_t^{2t} |h_t(r)|^2 \int_{\Sigma_\rho} |\widehat{f}(r\xi')|^2 \frac{d\sigma(\xi')}{\langle P\xi', n(\xi')\rangle} r^{\nu-1} dr\Big)^{1/2} \\
&\lesssim \Big(\int_t^{2t} |h_t(r)|^2 \|\tfrac{1}{r^\nu} f(\tfrac{\cdot}{r})\|_p^2 r^{\nu-1} dr\Big)^{\frac12} \\
&= \|f\|_p\Big(\int_t^{2t} |h_t(r)|^2 r^{2\nu(\frac1p-\frac12)} \frac{dr}{r}\Big)^{\frac12}.
\end{aligned}$$

Now, dropping the support assumption we decompose $h = \sum_{k\in\mathbb{Z}} h_{2^k}$ where h_{2^k} is supported in $[2^k, 2^{k+1}]$. Let

$$A = \sup_k 2^{k\nu(\frac{1}{p}-\frac{1}{2})}\Big(\int_{2^{k-1}}^{2^{k+1}} |h(\rho)|^2 \frac{d\rho}{\rho}\Big)^{1/2}.$$

Let $\varphi \in C^\infty$ supported in $(1/4, 4)$ so that $\varphi(s) = 1$ on $[1/2, 2]$. Define L_k by $\widehat{L_k f}(\xi) = \varphi(2^{-k}\rho(\xi))\widehat{f}(\xi)$. By Littlewood-Paley theory (adapted to the geometry determined by P) we know that the operator $f \mapsto \{L_k f\}_{k\in\mathbb{Z}}$ is bounded from L^p to $L^p(\ell^2)$. The above shows

$$\begin{aligned}\|\mathcal{F}^{-1}[h(\rho(\cdot))\widehat{f}]\|_2^2 &\lesssim \sum_k \|h_{2^k}\circ\rho\,\widehat{f}]\|_2^2 \\ &\lesssim A^2\sum_k \|L_k f\|_p^2 \lesssim A^2\Big\|\Big(\sum_k |L_k f|^2\Big)^{1/2}\Big\|_p^2 \lesssim A^2\|f\|_p^2,\end{aligned}$$

$\square$

Remark 2.3. It can be shown that for $\rho(\xi) = |\xi|^a$, and the operator acting on *radial functions* only we have the same inequality in a larger range, namely $L^p_{rad} \to L^2$ boundedness for $1 < p < \frac{2d}{d+1}$. Moreover at the endpoint $p_0 = 2d/(d+1)$, we have $L^{p_0,1}_{rad} \to L^2$ boundedness and $L^{p_0,1}$ cannot be replaced with $L^{p_0,q}_{rad}$ for $q > 1$. See [18].

3. An $\mathcal{F}L^q$ result

In this section h is assumed to be supported in a compact subinterval J of $(0,\infty)$. In order for $h\circ\rho$ to belong to M^q_p it is necessary for $\mathcal{F}^{-1}[h\circ\rho]$ to belong to L^q; this is immediate because on can test the convolution operator on Schwartz-functions whose Fourier transform is equal to one on the support of $h\circ\rho$. It is sometimes useful to express this condition in terms of a condition on the inverse Fourier transform of h (considered as a function on $(-\infty,\infty)$), i.e.

$$\kappa(r) = \int h(\rho)e^{-i\rho r}d\rho\,. \tag{3.1}$$

Such a characterization is possible in the isotropic case, when ρ is the Minkowski functional of an open set Ω with smooth boundary, starlike with respect to the origin in its interior, so that ρ is homogeneous of degree one and Σ_ρ is the boundary of Ω. The following result can be found for $\rho(\xi) = |\xi|$ in [18]. The proof for general ρ is taken from our joint paper with Lee [32].

Theorem 3.1. *Let $1 \le q \le 2$, Ω as above, let ρ be the Minkowski functional of Ω and let J be a compact subinterval of $(0, \infty)$. Then for h with support in J ,*

$$\big\|\mathcal{F}^{-1}[h \circ \rho]\big\|_{L^q(\mathbb{R}^d)} \approx \Big(\int |\kappa(r)|^q (1+|r|)^{(d-1)(1-\frac{q}{2})} dr \Big)^{1/q}. \tag{3.2}$$

The implicit constant depends on J.

Proof. Let $\chi \in \mathcal{S}(\mathbb{R})$ so that χ is compactly supported in $(0, \infty)$ and $\chi(s) = 1$ on $\operatorname{supp}(h)$. The inequality "$\lesssim$" in (3.2) follows by showing

$$\begin{aligned} &\Big\|\mathcal{F}^{-1}\big[\chi(\rho(\cdot)) \int \kappa(r) e^{ir\rho(\cdot)} dt\big]\Big\|_{L^q(\mathbb{R}^d)} \\ &\qquad\qquad \lesssim \Big(\int |\kappa(r)|^q (1+|r|)^{(d-1)(1-\frac{q}{2})} dr \Big)^{1/q} \end{aligned} \tag{3.3}$$

for the cases $q = 2$ and $q = 1$, and using analytic interpolation.

For $q = 2$ we have

$$\begin{aligned} &(2\pi)^d \Big\|\mathcal{F}^{-1}[\chi(\rho(\cdot)) \int \kappa(r) e^{ir\rho(\cdot)} dr]\Big\|^2_{L^2(\mathbb{R}^d)} \\ &= \int \Big|\chi(\rho(\xi)) \int \kappa(r) e^{ir\rho(\xi)} dr\Big|^2 d\xi \\ &= c \int \Big|\chi(\rho) \int \kappa(r) e^{ir\rho} dr\Big|^2 \rho^{d-1} d\rho \\ &\lesssim \int |\kappa(r)|^2 dr \end{aligned}$$

where we have first used Plancherel in $\mathbb{R}^d$, applied generalized polar coordinates, and then used Plancherel on the real line.

In order to prove (3.3) for $q = 1$ we use the fact

$$\Big\|\mathcal{F}^{-1}[\chi(\rho(\cdot)) e^{ir\rho(\cdot)}]\Big\|_{L^1(\mathbb{R}^d)} \lesssim (1+|r|)^{\frac{d-1}{2}}. \tag{3.4}$$

This is a rescaled version of an inequality in [44]; the assumption that ρ is homogeneous of degree one is crucial here. (3.4) and Minkowski's integral inequality immediately give

$$\Big\|\mathcal{F}^{-1}[\chi(\rho(\cdot)) \int \kappa(r) e^{ir\rho(\cdot)} dt]\Big\|_1 \lesssim \int |\kappa(r)| (1+|r|)^{\frac{d-1}{2}} dr$$

which is (3.3) for $q = 1$.

We now show the converse inequality

$$\Big(\int |\kappa(r)|^q (1+|r|)^{(d-1)(1-q/2)} dr \Big)^{1/q} \lesssim \|\mathcal{F}^{-1}[h \circ \rho]\|_{L^q(\mathbb{R}^d)}. \tag{3.5}$$

for h smooth. This a priori assumption implies that the left hand side in (3.5) is finite; it is easy to remove by an approximation argument.

Let $\xi_0 \in \Sigma_\rho$ be a point where the Gaussian curvature does not vanish (e.g. a point where $|\xi|$ is maximal on Σ_ρ). Let U be a small neighborhood of ξ_0 on which the Gauss map is injective and the curvature is bounded below. Let γ be homogeneous of degree zero, $\gamma(\xi_0) \neq 0$ and supported on the closure of the cone generated by U. Then

$$\big\|\mathcal{F}^{-1}[\gamma\, h\circ\rho]\big\|_{L^q} \lesssim \big\|\mathcal{F}^{-1}[h\circ\rho]\big\|_{L^q}.$$

Use polar coordinates (with respect to ρ) to write

$$(2\pi)^d\mathcal{F}^{-1}[\gamma\, h\circ\rho](x) = \int_0^\infty h(\rho)\rho^{d-1}\int_{\Sigma_\rho}\gamma(\xi')e^{i\rho\langle\xi',x\rangle}\frac{d\sigma(\xi')}{|\nabla\rho(\xi')|}\,d\rho\,. \tag{3.6}$$

Let $n(\xi_0)$ the outer normal at ξ_0, let $\Gamma = \{x\in\mathbb{R}^d : \big|\frac{x}{|x|} - n(\xi_0)\big| \le \varepsilon\}$, with ε small and let, for large $R \gg 1$, $\Gamma_R = \{x\in\Gamma : |x| \ge R\}$. We may assume that for each $x\in\Gamma$ there is a unique $\xi = \Xi(x)\in\Sigma_\rho$, so that $\gamma(\Xi(x)) \neq 0$ and so that x is normal to Σ_ρ at $\Xi(x)$. Clearly $x\mapsto\Xi(x)$ is homogeneous of degree zero on Γ. By the method of stationary phase we have for $x\in\Gamma_R$

$$\mathcal{F}^{-1}[\gamma h(\rho(\cdot))](x) = I_0(x) + \sum_{j=1}^N II_j(x) + III(x) \tag{3.7}$$

where

$$I_0(x) = c\int_0^\infty h(\rho)\rho^{d-1}e^{i\rho\langle\Xi(x),x\rangle}d\rho\ \frac{\gamma(\Xi(x))|\nabla\rho(\Xi(x))|^{-1}}{(\rho\langle\Xi(x),x\rangle)^{\frac{d-1}{2}}|K(\Xi(x))|^{1/2}}$$

where $K(\Xi(x))$ is the Gaussian curvature at $\Xi(x)$ and $|c| = (2\pi)^{-d}$. There are similar formulas for the higher order terms $II_j(x)$, with the main term $(\rho\langle\Xi(x),x\rangle)^{-\frac{d-1}{2}}$ replaced by $(\rho\langle\Xi(x),x\rangle)^{-\frac{d-1}{2}-j}$. Finally

$$|III(x)| \lesssim_N \|h\|_1|x|^{-N}\,, \quad x\in\Gamma_R\,.$$

Let $h_j(\rho) = h(\rho)\rho^{\frac{d-1}{2}-j}$ and let $\kappa_j = \mathcal{F}_{\mathbb{R}}^{-1}[h_j]$, then

$$|I_0(x)| \approx \Big|\frac{\kappa_0(\langle\Xi(x),x\rangle)}{\langle\Xi(x),x\rangle^{\frac{d-1}{2}}}\Big|, \quad x\in\Gamma_R.$$

We also have

$$|x| \approx \langle\Xi(x),x\rangle, \quad x\in\Gamma,$$

which is a consequence of $\rho(\xi) = \langle \xi, \nabla\rho(\xi)\rangle$ (Euler's homogeneity relation) and the positivity of ρ on Σ_ρ. It follows

$$\|I_0\|_{L^q(\Gamma_R)} \gtrsim \Big(\int_{C_0R}^\infty |\kappa_0(r)|^q (1+|r|)^{(d-1)(1-q/2)} dr \Big)^{1/q} \tag{3.8}$$

for some $C_0 > 1$. Similarly

$$\|II_j\|_{L^q(\Gamma_R)} \lesssim R^{-j} \Big(\int_0^\infty |\kappa_j(r)|^q (1+|r|)^{(d-1)(1-q/2)} dr \Big)^{1/q} . \tag{3.9}$$

We will need to use the straightforward bound

$$\Big(\int_0^\infty |\zeta * g(r)|^q (1+|r|)^a dr \Big)^{1/q} \lesssim \Big(\int_0^\infty |g(r)|^q (1+|r|)^a dr \Big)^{1/q} \tag{3.10}$$

whenever $\zeta \in \mathcal{S}(\mathbb{R})$. Since $\kappa_j = \zeta_j * \kappa_0$ this shows that we can replace κ_j in (3.9) by κ_0.

There are also the trivial inequalities

$$\|III\|_{L^q(\Gamma_R)} \lesssim R^{-1} \|h\|_1$$

and

$$\|h\|_1 \lesssim \|h\circ\rho\|_{L^{q'}(\mathbb{R}^d)} \lesssim \|\mathcal{F}^{-1}[h\circ\rho]\|_{L^q(\mathbb{R}^d)} .$$

Thus

$$\|III\|_{L^q(\Gamma_R)} \lesssim R^{-1} \|\mathcal{F}^{-1}[h\circ\rho]\|_{L^q(\mathbb{R}^d)}. \tag{3.11}$$

Moreover, we estimate crudely

$$\Big(\int_0^{C_0R} |\kappa(r)|^q (1+|r|)^{(d-1)(1-q/2)} dr \Big)^{1/q} \lesssim R^{d/q} \|\kappa_0\|_\infty \lesssim R^{d/q} \|h\|_1 . \tag{3.12}$$

We now combine the estimates and get

$$\begin{aligned} &\Big(\int |\kappa_0(r)|^2 (1+|r|)^{(d-1)(1-q/2)} dr \Big)^{1/q} \\ &\lesssim \|I_0\|_{L^q(\Gamma_R)} + R^{d/q} \|\mathcal{F}^{-1}[h\circ\rho]\|_{L^q(\mathbb{R}^d)} \\ &\lesssim R^{d/q} \|\mathcal{F}^{-1}[h\circ\rho]\|_{L^q(\mathbb{R}^d)} + R^{-1} \sum_{j=1} \|II_j\|_{L^q(\mathbb{R}^d)} \end{aligned} \tag{3.13}$$

here we have used (3.7), (3.11)) for the second inequality, and (3.8), (3.12)) for the third. To estimate the second term in (3.13) we use (3.9) and then, using (3.10) replace κ_j with κ_0 to get

$$\|II_j\|_{L^q(\Gamma_R)} \lesssim R^{-1} \Big(\int_0^\infty |\kappa_0(r)|^q (1+|r|)^{(d-1)(1-q/2)} dr \Big)^{1/q} .$$

Thus, choosing R sufficiently large (and using the finiteness of the right hand side of the last display) we get

$$\Big(\int |\kappa_0(r)|^2(1+|r|)^{(d-1)(1-q/2)}dr\Big)^{1/q} \lesssim \|\mathcal{F}^{-1}[h\circ\rho]\|_{L^q(\mathbb{R}^d)}.$$

Finally observe that $\kappa = \kappa_0 * \zeta_0$ for some Schwartz function ζ_0 and thus, by (3.10) we may replace κ_0 by κ. This finishes the proof of (3.5). $\square$

Remark. An application of the Hausdorff-Young inequality recovers necessary comditions in [19],[42], namely for h supported in a compact interval $J \subset (0,\infty)$

$$\|h\|_{B^{q'}_{(d-1)(\frac1q-\frac12),q}} \lesssim_J \|\mathcal{F}^{-1}[h\circ\rho]\|_{L^q(\mathbb{R}^d)}. \tag{3.14}$$

In [42] inequality (3.14) was proved for more general P-homogeneous distance functions ρ, but we don't have a simple analogue of Theorem 3.1 in this case.

4. Radial M_p multipliers acting on radial functions

Let g be a radial Schwartz-function, $g = g_0(|x|)$ then the Fourier transform of g is given by $\widehat{g}(\xi) = \mathcal{B}_d g_0(\rho)$ where $\mathcal{B}_d$ denotes the Fourier-Bessel transform (or modified Hankel transform) of g_0. It is given by

$$\mathcal{B}_d f(\rho) = \int_0^\infty f(s)\mathfrak{J}(s\rho)s^{d-1}ds \tag{4.1}$$

where

$$\mathfrak{J}(\rho) = \rho^{-\frac{d-2}{2}} J_{\frac{d-2}{2}}(\rho) \tag{4.2}$$

and J_α denotes the standard Bessel function.

The convolution with a radial kernel maps radial functions to radial functions. Let $h(|\xi|)$ be a radial multiplier and $g = g_0(|\cdot|)$ Then

$$\mathcal{F}^{-1}[h(|\cdot|)\widehat{g}](x) \,=\, c\,\mathcal{B}_d[h\mathcal{B}_d g_0](|x|)\,. \tag{4.3}$$

We denote by L^p_{rad} the space of radial L^p functions. The following characterization of L^p_{rad} boundedness was proved in joint work with G. Garrigós [18]. Let ϕ be a nontrivial bump function which is compactly supported in $(0,\infty)$.

Theorem 4.1. *Let $1 < p < \frac{2d}{d+1}$. Define $\kappa_t(\tau) = \mathcal{F}^{-1}[\phi h(t\cdot)](\tau)$. Then*

$$\|\mathcal{F}^{-1}[h(|\cdot|)\widehat{f}]\|_{L^p_{rad}} \lesssim \|f\|_{L^p_{rad}}$$

for all radial Schwartz functions holds if and only if the condition

$$\sup_{t>0}\Big(\int|\kappa_t(\tau)|^p(1+|\tau|)^{(d-1)(1-p/2)}d\tau\Big)^{1/p}<\infty \tag{4.4}$$

is satisfied.

The necessity is easily checked by testing the operator on functions of the form $t^{d/p}\eta(t\cdot)$ where η is a suitable Schwartz function. For the full proof of Theorem 4.1 we refer to [18]. Here we consider only the special case where h is compactly supported on a compact subinterval J of $(0,\infty)$. In this case the condition (4.4) reduces to the finiteness of $\|\mathcal{F}^{-1}[h(|\cdot|)]\|_p$ and thus, by Theorem 3.1, we need to prove the estimate

$$\begin{aligned}\big\|\mathcal{B}_d[h\mathcal{B}_d g]\big\|_{L^p(r^{d-1}dr}\\ \lesssim\Big(\int_{-\infty}^{\infty}|\widehat{h}(\tau)|^p(1+|\tau|)^{(d-1)(1-p/2)}d\tau\Big)^{1/p}\|g\|_{L^p(r^{d-1}dr)}\,.\end{aligned}\tag{4.5}$$

Proof of (4.5). Here we give the simple idea for the proof of this special case. We need to use the standard asymptotics for Bessel functions (see [14], 7.13.1(3)), namely for $|x|\ge 1$,

$$\begin{aligned}\mathfrak{J}(x)=\sum_{\nu=0}^{M}c_{\nu,d}\cos(x-\tfrac{d-1}{4}\pi)x^{-2\nu-\frac{d-1}{2}}\\ +\sum_{\nu=0}^{M}\widetilde{c}_{\nu,d}\sin(x-\tfrac{d-1}{4}\pi)x^{-2\nu-\frac{d+1}{2}}+x^{-M}\widetilde{E}_{M,d}(x)\end{aligned}\tag{4.6}$$

with $c_{0,d}=(2/\pi)^{1/2}$, and the derivatives of $\widetilde{E}_{M,d}$ are bounded.

Note that

$$\mathcal{B}_d[h\mathcal{B}_d g](r)=\int K(r,s)s^{d-1}g(s)ds \tag{4.7}$$

where

$$K(r,s)=\int_0^\infty h(\rho)\mathfrak{J}(\rho r)\mathfrak{J}(\rho s)\rho^{d-1}d\rho. \tag{4.8}$$

Using the above asymptotic expansion in this integral, for both $\mathfrak{J}(\rho r)$ and $\mathfrak{J}(\rho s)$ one derives the following pointwise bound

$$|K(r,s)|\lesssim\sum_{(\pm,\pm)}(1+r)^{-\frac{d-1}{2}}(1+s)^{-\frac{d-1}{2}}\int\frac{|\kappa(\pm r\pm s-u)|}{(1+|u|)^N}du. \tag{4.9}$$

where $\kappa = \mathcal{F}_{\mathbb{R}}^{-1}[h]$. If we incorporate the weights into the operator, set $f(s) = s^{(d-1)/p} g(s)$, and use (4.9), then (4.5) reduces to

(4.10)
$$\Big(\int \Big[\frac{|r|^{\frac{d-1}{p}}}{(1+|r|)^{\frac{d-1}{2}}} \iint \frac{|s|^{\frac{d-1}{p'}}}{(1+|s|)^{\frac{d-1}{2}}} \frac{|\kappa(\pm r \pm s - u)|}{(1+|u|)^N} |f(s)|\, ds du\Big]^p dr\Big)^{1/p} \\ \lesssim \|\kappa\|_{L^p((1+|r|)^{(d-1)(1-p/2)}dr)} \|f\|_{L^p(dr)}\,.$$

Let
$$\mathcal{K}(r,s) = \Big(\frac{1+|r|}{1+|s|}\Big)^{(d-1)(\frac{1}{p}-\frac12)} |\kappa(\pm r \pm s)|$$

By an application of Minkowski's inequality and a straightforward estimate we see that (4.10) follows from

$$\Big(\int \Big|\int \mathcal{K}(r,s) f(s) ds\Big|^p dr\Big)^{1/p} \\ \lesssim \|\kappa\|_{L^p((1+|\tau|)^{(d-1)(1-p/2)}d\tau)} \|f\|_{L^p(ds)}\,. \tag{4.11}$$

We split the kernel into $\mathcal{K}^1(r,s) = \mathcal{K}(r,s)\chi_{|s|\le |r|/2}$ and $\mathcal{K}^2(r,s) = \mathcal{K}(r,s)\chi_{|s|\ge |r|/2}$. The main contribution comes from the first kernel and we get

$$\Big(\int \Big|\int \mathcal{K}^1(r,s) f(s) ds\Big|^p dr\Big)^{1/p} \\ \lesssim \int \frac{|f(s)|}{(1+|s|)^{(d-1)(\frac1p-\frac12)}} \Big(\int_{|r|\ge 2|s|} (1+|r|)^{(d-1)(1-\frac{p}{2})} |\kappa(\pm r \pm s)|^p dr\Big)^{1/p} ds \\ \lesssim \int \frac{|f(s)|}{(1+|s|)^{(d-1)(\frac1p-\frac12)}} ds\, \Big(\int (1+|r|)^{(d-1)(1-\frac{p}{2})} |\kappa(r)|^p dr\Big)^{1/p}.$$

For $p < \frac{2d}{d+1}$ Hölder's inequality yields
$$\int (1+|s|)^{-(d-1)(\frac12-\frac1p)} |f(s)| ds \lesssim \|f\|_p$$

and we get the analogue of (4.11) for the kernel $\mathcal{K}^1$.

For the contribution with $|s| \ge |r|/2$ we can drop the term
$$\Big(\frac{1+|r|}{1+|s|}\Big)^{(d-1)(1/p-1/2)} \lesssim 1$$

and estimate
$$\Big(\int \Big|\int \mathcal{K}^2(r,s) f(s) ds\Big|^p dr\Big)^{1/p} \lesssim \|\kappa\|_1 \|f\|_p\,.$$

Since by Hölder's inequality and $p < \frac{2d}{d+1}$

$$\|\kappa\|_1 \lesssim \Big(\int (1+|r|)^{(d-1)(1-\frac{p}{2})} |\kappa(r)|^p dr \Big)^{1/p},$$

we get the analogue of (4.11) for the kernel $\mathcal{K}^2$. This finishes the proof of (4.5). $\square$

Open problem: Is there an effective characterization of $L^p_{rad} \to L^p_{rad}$ boundedness in the range $\frac{2d}{d+1} \le p < 2$?

5. Radial Fourier multipliers with compact support

We now give the proof of the characterization of radial multipliers with compact support away from the origin, which was obtained in the paper with Heo and Nazarov [23].

Theorem 5.1. *Let $d \ge 4$, $1 < p < p_d := \frac{2d-2}{d+1}$, and let m be radial,*

$$m = h(|\cdot|)$$

with h supported in $(\frac12, 2)$. Let κ as in (3.1), i.e. $\kappa(r) = (2\pi)\mathcal{F}^{-1}[h](r)$.

The following are equivalent:

(i) $m \in M^p(\mathbb{R}^d)$

(ii) $\Big(\int_{-\infty}^{\infty} |\kappa(r)|^p (1+|r|)^{(d-1)(1-p/2)} dr \Big)^{1/p} < \infty$

(iii) $\mathcal{F}^{-1}[m] \in L^p(\mathbb{R}^d)$.

For the equivalence of (ii) and (iii) see §3. Clearly (i) implies (iii), in view of the compact support of m. In this section we prove that (iii) implies (i).

Let ϕ_0 be a radial Schwartz function, compactly supported in $\{|x| \le \frac12\}$ such that $\widehat{\phi_0}(\xi) > 0$ on $\{\xi : 1/2 \le |\xi| \le 2\}$. Take $\psi_0 = \Delta^{2M}\phi_0$ for $M > 5d$. Then ψ_0 is a radial Schwartz function, compactly supported in $\{|x| \le \frac12\}$ such that $\widehat{\psi_0}(\xi) > 0$ on $\{\xi : 1/2 \le |\xi| \le 2\}$ and such that

$$\int \psi_0(x) P(x) dx = 0$$

for all polynomials of degree $\le 4M$. Let

$$\psi = \psi_0 * \psi_0.$$

Let η be a radial Schwartz function such that $\widehat{\eta}(\xi) = [\widehat{\psi}(\xi)]^{-1}$ for $1/2 \le |\xi| \le 2$. Then $h(|\xi|)\widehat{\psi}(\xi)\widehat{\eta}(\xi) = m(\xi)$ and thus

$$\mathcal{F}^{-1}[m] = \psi * K = \psi_0 * \psi_0 * K$$

where $K = \eta * \mathcal{F}^{-1}[m]$. Note that K is radial and $\widehat{K}$ is compactly supported in $\{\xi : 1/2 \le |\xi| \le 2\}$. Clearly

$$\|K\|_p \lesssim \|\mathcal{F}^{-1}[m]\|_p \,.$$

Thus we need to show

$$\|\psi * K * f\|_p \lesssim \|K\|_p \|f\|_p. \tag{5.1}$$

Let $\mathcal{K}(r)$ be defined on $(0,\infty)$ so that $\mathcal{K}(|x|) = K(x)$.

If $g \in \mathcal{S}$ we have, by polar coordinates,

$$\int K(x)g(x)dx = \int_0^\infty \mathcal{K}(r)r^{d-1} \int_{S^{d-1}} g(rx')d\sigma\, dr$$

and thus

$$K = \int_0^\infty \mathcal{K}(r)\sigma_r dr \tag{5.2}$$

where σ_r is the surface measure on the sphere of radius r and (5.2) is understood in the sense of distributions. Note that

$$\|\sigma_r\|_M = O(r^{d-1}).$$

Also $\langle \sigma_r, g\rangle = r^{d-1}\langle \sigma, g(r\cdot)\rangle$, i.e. $\sigma_r = r^{-1}\sigma(r^{-1}\cdot)$.

Lemma 5.2. *We have (*

$$|\widehat{\sigma}_r(\xi)| \le \frac{r^{d-1}}{(1+|\xi|r)^{\frac{d-1}{2}}}$$

and

$$\|\widehat{\psi_0 * \sigma_r}\|_\infty \lesssim (1+r)^{\frac{d-1}{2}} \tag{5.3}$$

Proof. The first formula follows from stationary phase or the explicit formula ([48])

$$\widehat{\sigma}_1(|\xi|) = \mathcal{J}(|\xi|), \tag{5.4}$$

with $\mathcal{J}$ as in (4.2)

Now (5.3) follows because $\widehat{\psi}$ vanishes at the origin to order $20d$. $\square$

The contribution of the integral in (5.2) for $r \le A$ is trivial for our purposes since

$$\Big\| \int_0^A \mathcal{K}(r)\sigma_r dr \Big\|_1 \lesssim \int_0^A |\mathcal{K}(r)| r^{d-1} dr$$
$$\lesssim A^{d/p'} \Big(\int_0^A |\mathcal{K}(r)|^p r^{d-1} dr \Big)^{1/p}.$$

Let $f \in L^p(\mathbb{R}^d)$. We need to prove

$$\Big\| \int_2^\infty \mathcal{K}(r) \int \psi * \sigma_r(x-y) f(y) dy\, dr \Big\|_p \lesssim \Big(\int_2^\infty |\mathcal{K}(r)|^p r^{d-1} dr \Big)^{1/p} \|f\|_p. \tag{5.5}$$

Let

$$F_{y,r}(x) := \psi * \sigma_r(x-y). \tag{5.6}$$

We replace the tensor product function $f(y)\mathcal{K}(r)$ with a more general function g in the Lebesgue space $L^p(\mathbb{R}^d \times (0,\infty); dy\, r^{d-1} dr)$. We will then prove the more general inequality

$$\Big\| \int_2^\infty \int F_{y,r}(x) g(y,r) dy\, dr \Big\|_p \lesssim \Big(\int_2^\infty \int |g(y,r)|^p dy\, r^{d-1} dr \Big)^{1/p} \tag{5.7}$$

It is convenient for notation to discretize the above inequality. This is natural in view of the compact support of the Fourier transform of K and (w.l.o.g.) the Fourier transform of f. For $u = (u', u_{d+1}) \in [0,1]^d \times [0,1]$ let $\mathcal{Z}_u$ be the half-lattice consisting of those $(y,r) = (y_1, \dots, y_d, r)$ such that $y_i = z_i + u_i$ for some integer z_i and $r = n + u_{d+1}$ for some integer $n \ge 2$. It suffices to prove the inequality

$$\Big\| \sum_{(y,r)\in\mathcal{Z}_u} F_{y,r}(x) g(y,r) \Big\|_p \lesssim \Big(\sum_{(y,r)\in\mathcal{Z}_u} |g(y,r)|^p\, r^{d-1} \Big)^{1/p} \tag{5.8}$$

with a bound uniform in $u \in [0,1]^{d+1}$. Henceforth it is assumed that all (y,r) sums are taken over $\mathcal{Z}_u$. Later we shall also use the notation

$$\mathcal{Z}_{k,u} = \{(y,r) \in \mathcal{Z}_u : 2^k \le r < 2^{k+1}\}. \tag{5.9}$$

Inequality (5.7) (with the r-integration over $[1,\infty)$) is an immediate consequence of (5.8), by averaging and Hölder's inequality. Indeed, we have

$$\begin{aligned}
&\Big\|\int_1^\infty\int \psi*\sigma_r(x-y)g(y,r)dy\,dr\Big\|_p\\
&\lesssim \Big\|\int_{[0,1]^{d+1}}\sum_{(y,r)\in\mathcal{Z}_u}\psi*\sigma_r(x-y)g(y,r)\Big\|_p du\\
&\lesssim \int_{[0,1]^{d+1}}\Big\|\sum_{(y,r)\in\mathcal{Z}_u}\psi*\sigma_r(x-y)g(y,r)\Big\|_p du\\
&\lesssim \Big(\int_{[0,1]^{d+1}}\Big\|\sum_{(y,r)\in\mathcal{Z}_u}F_{y,r}(x)g(y,r)\Big\|_p^p du\Big)^{1/p}
\end{aligned}$$

and, once we prove (5.8), the right hand side of the previous display is bounded by a constant times

$$\Big(\int_{[0,1]^{d+1}}\sum_{(y,r)\in\mathcal{Z}_u}|g(y,r)|^p\,r^{d-1}\,du\Big)^{1/p}=\Big(\int_1^\infty\int|g(y,r)|^p dy\,r^{d-1}dr\Big)^{1/p}.$$

Support properties. Recall that the support of ψ is in the unit ball and thus

$$\operatorname{supp}(F_{y,r})\subset\{x:\big||x-y|-r\big|\le 1\},\tag{5.10}$$

which has measure $O(r^{d-1})$ if $r\ge 2$. Thus if $\mathcal{E}$ is a subset of $\mathcal{Z}_u$, $\mathcal{Z}_{k,u}$ as in (5.9), then we have the trivial consequence

$$\operatorname{meas}\Big(\operatorname{supp}\big(\sum_k\sum_{(y,r)\in\mathcal{E}\cap\mathcal{Z}_{k,u}}F_{y,r}\big)\Big)\lesssim\sum_k 2^{k(d-1)}\#(\mathcal{E}\cap\mathcal{Z}_{k,u})\,.\tag{5.11}$$

This estimate may be improved if the set $\mathcal{E}\subset\mathcal{Z}_{k,u}$ is concentrated on a ball of radius $R_0\in(1,2^k)$. Observe that the cardinality of $\mathcal{E}$ is always $O(R_0^{d+1})$ in this case and an improvement over (5.11) happens if this cardinality is substantially larger than R_0.

Lemma 5.3. *Let $1\le R_0\le 2^k$, $\mathcal{E}_k\subset\mathcal{Z}_{k,u}$ and suppose that $\mathcal{E}_k$ is contained in a ball B_0 of radius R_0. Then*

$$\operatorname{meas}\Big(\operatorname{supp}\big(\sum_{(y,r)\in\mathcal{E}_k}|F_{y,r}|\big)\Big)\lesssim 2^{k(d-1)}R_0\,.\tag{5.12}$$

Proof. If (y_0,r_0) is the center of B_0 and if $F_{y,r}(x)\ne 0$ then $\big||x-y|-r\big|\le 1$ and consequently

$$\big||x-y_0|-r_0\big|\le\big||x-y|-r\big|+|y-y_0|+|r-r_0|\le 1+2R_0.$$

Thus

$$\operatorname{supp}\Big(\sum_{(y,r)\in\mathcal{E}_k} F_{y,r}\Big) \subset \{(y,r)\in\mathcal{Z}_{k,u} : \big||x-y_0|-r_0\big| \le 2R_0+1\}$$

and the measure of this set is $O(2^{k(d-1)}R_0)$. □

A weak orthogonality property. Inequality (5.8) trivially holds for $p=1$ since $\|F_{y,r}\|_1 = O(r^{d-1})$. We also know that $\|F_{y,r}\|_2^2 = O(r^{d-1})$ and thus if the functions $F_{y,r}$ were orthogonal, or almost orthogonal in a strong sense then one would get the inequality for $p=2$. However, this would imply that we get L^2 boundedness for convolution operators for which the compactly supported Fourier multiplier is merely in L^2, but of course boundedness of the Fourier multiplier is a necessary and sufficient condition for L^2 boundedness. Consequently the $F_{y,r}$ are not "almost orthogonal" enough. Nevertheless there is a weak orthogonality which will be crucial in our estimates. It is expressed in the following

Lemma 5.4.

$$\big|\langle F_{y,r}, F_{y',r'}\rangle\big| \lesssim \frac{(rr')^{\frac{d-1}{2}}}{(1+|y-y'|+|r-r'|)^{\frac{d-1}{2}}}. \tag{5.13}$$

Proof. By Parsevals's identity

$$\langle F_{y,r}, F_{y',r'}\rangle = (2\pi)^{-d}\langle \widehat{F}_{y,r}, \widehat{F}_{y',r'}\rangle$$

Recall that $\widehat{\sigma}_1(|\xi|) = \mathcal{J}(|\xi|)$ (*cf.* (5.4)) and thus, with $\widehat{\psi}(\xi)=\beta(|\xi|)$, we have

$$\begin{aligned}
&(2\pi)^d\langle \widehat{F}_{y,r}, \widehat{F}_{y',r'}\rangle \\
&= \int \widehat{\sigma}_r(\xi)\widehat{\sigma}_{r'}(\xi)|\widehat{\psi}(\xi)|^2 e^{i\langle y'-y,\xi\rangle}d\xi \\
&= (rr')^{d-1}\int |\beta(\rho)|^2\mathcal{J}(r\rho)\mathcal{J}(r'\rho)\int_{S^{d-1}} e^{i\rho\langle y'-y,\xi'\rangle}d\sigma(\xi')\,\rho^{d-1}d\rho \\
&= (rr')^{d-1}\int |\beta(\rho)|^2\rho^{d-1}\mathcal{J}(r\rho)\mathcal{J}(r'\rho)\mathcal{J}(\rho|y-y'|)\,d\rho
\end{aligned}$$

Now $|\mathcal{J}(s)| \lesssim (1+|s|)^{-\frac{d-1}{2}}$ and if we take into account that $|\beta(s)| \lesssim |s|^{20d}$ for $|s|\le 1$ and $|\beta(s)|\lesssim |s|^{-N}$ for $|s|\ge 1$, we obtain

$$\big|\langle F_{y,r}, F_{y',r'}\rangle\big| \lesssim \frac{(rr')^{\frac{d-1}{2}}}{(1+|y-y'|)^{\frac{d-1}{2}}}.$$

This gives the asserted bound when $|r-r'| \le C(1+|y-y'|)$. But from

$$\begin{aligned} &\mathrm{supp}(F_{y,r}) \subset \{x : \big||x-y|-r\big| \le 1\}, \\ &\mathrm{supp}(F_{y',r'}) \subset \{x : \big||x-y'|-r'\big| \le 1\} \end{aligned}$$

we see that for $|r-r'| \gg 1+|y-y'|$, the supports of $F_{y,r}$ and $F_{y',r'}$ are disjoint. Hence $\langle F_{y,r}, F_{y',r'}\rangle = 0$ in this case. $\square$

We note that one can prove a finer estimate

$$|\langle F_{y,r}, F_{y',r'}\rangle| \le C_N (rr')^{\frac{d-1}{2}} (1+|y-y'|)^{-\frac{d-1}{2}} \sum_{\pm,\pm} \big(1+\big|r \pm r' \pm |y-y'|\big|\big)^{-N} \tag{5.14}$$

if one uses the oscillations of the Bessel functions (as in (4.6). We will not have to use (5.14).

Reduction to a restricted weak type inequality. We need to prove the inequality (5.8) for p in the open range $(1, \frac{2(d-1)}{d+1})$. Consider the operator $\mathcal{T}$ acting on functions $\{g(y,r)\}$ on the lattice $\mathcal{Z}_u$, defined by

$$\mathcal{T} g(x) = \sum_{(y,r)} F_{y,r}(x)\, g(y,r)\,.$$

By the generalized Marcinkiewicz interpolation theorem it suffices to prove that $\mathcal{T}$ maps the weighted Lorentz space $\ell^{p,1}(\mathcal{Z}_u, r^{d-1})$ to $L^{p,\infty}(\mathbb{R}^d)$. For $k = 1, 2, \dots$ let

$$\mathcal{E}_k \subset \mathcal{Z}_{k,u} \tag{5.15}$$

and let $c_{y,r}$ be constants satisfying

$$\sup_{y,r} |c_{y,r}| \le 1\,. \tag{5.16}$$

This $\ell^{p,1}(\mathcal{Z}_u, r^{d-1}) \to L^{p,\infty}(\mathbb{R}^d)$ bound follows if we can show the restricted weak type inequality

$$\mathrm{meas}\Big(\Big\{x \in \mathbb{R}^d : \Big|\sum_k \sum_{(y,r)\in\mathcal{E}_k} c_{y,r} F_{y,r}\Big| > \lambda\Big\}\Big) \lesssim \lambda^{-p} \sum_k 2^{k(d-1)} \#\mathcal{E}_k\,. \tag{5.17}$$

We note that (5.17) is true for $p = 1$, and thus implies (5.17) for $p > 1$ when $\lambda \le 1$. Hence in what follows we will assume that $\lambda > 1$. We may also assume that $\#\mathcal{E}_k < \infty$ for all k.

Modified Calderón-Zygmund decompositions. The following proposition is motivated by the support estimate in Lemma 5.3. In what follows $\lambda > 1$.

Proposition 5.5. *For every k there is finite collection $\mathfrak{B}_k$ of disjoint balls so that*

(i) Each ball $\mathcal{B} \in \mathfrak{B}_k$ has radius $rad(B) \le 2^k$ and

$$\#(B \cap \mathcal{E}_k) \ge \lambda^p \operatorname{diam}(B)\,.$$

(ii) For each $B \in \mathfrak{B}_k$ denote by B^* the ball with same center, and radius equal to five times the radius of B. Define the sparse set (or low density set) $\mathcal{E}_k^{\rm sp}$ as

$$\mathcal{E}_k^{\rm sp} = \mathcal{E}_k \setminus \bigcup_{B \in \mathfrak{B}_k} B^*.$$

Then for every subset D of $\mathcal{Z}_{k,u}$ with diameter $\le 2^{k+1}$ we have

$$\#(\mathcal{E}^{\rm sp} \cap D) \lesssim \lambda^p \operatorname{diam}(D)\,.$$

Proof. This is analogous to the proof of the usual Vitali type covering lemma. We set $\mathfrak{B}_{0,k} = \emptyset$. If there are no balls of radius at most 2^k with the property that the cardinality of the intersection is at least λ^p times the diameter of the ball then we set $\mathcal{E}_k^{\rm sp} = \mathcal{E}_k$ and properties (i) and (ii) are satisfied with $\mathfrak{B}_k = \mathfrak{B}_{0,k} = \emptyset$. Otherwise we choose a maximal ball $B_{1,k}$ of radius at most 2^k such that $\#(B_{1,k} \cap \mathcal{E}_k) \ge \lambda^p \operatorname{diam}(B_{1,k})\,.$

At stage ℓ we are given a collection $\mathfrak{B}_{\ell-1,k} = \{B_{1,k}, \dots, B_{\ell-1,k}\}$ of $\ell-1$ *disjoint* balls such that $\#(\mathcal{E}_k \cap B_{i,k}) \ge \lambda^p \operatorname{diam}(B_{i,k})$ for $i = 1, \dots, \ell-1$ and such that the radii of $B_{i,k}$ do not increase if i increases. If there are no balls of radius at most 2^k in the complement of $\cup_{i=1}^{\ell-1} B_{i,k}$ such that the cardinality of the intersection with $\mathcal{E}_k$ is at least λ^p times the diameter of the ball then we set $\mathcal{E}_k^{\rm sp} = \mathcal{E}_k \setminus \cup_{i=1}^{\ell-1} B_{1,k}^*$ and the construction stops. Otherwise choose a maximal ball $B_{\ell,k}$ of radius at most 2^k in the complement of $\cup_{i=1}^{\ell-1} B_{i,k}$ such that $\#(B_{\ell,k} \cap \mathcal{E}_k) \ge \lambda^p \operatorname{diam}(B_{\ell,k})$. Note that $\operatorname{diam}(B_{\ell,k}) \le \operatorname{diam}(B_{i.k})$ for $i = 1, \dots, \ell-1$ (since otherwise $B_{\ell,k}$ would have been selected before). We set $\mathfrak{B}_{k,\ell} = \{B_{1,k}, \dots, B_{\ell,k}\}$.

The construction stops at some stage, after having selected disjoint balls $B_{i,k}$, for $i = 1, \dots, N_k$. Then we set

$$\mathfrak{B}_k := \mathfrak{B}_{N_k,k} = \{B_{1,k}, \dots, B_{N_k,k}\}\,,$$
$$\mathcal{E}_k^{\rm sp} = \mathcal{E}_k \setminus \cup_{i=1}^{N_k} B_{i,k}^*\,.$$

If B is *any* ball satisfying $\#(B \cap \mathcal{E}_k) \ge \lambda^p \operatorname{diam}(B)$ then B must be contained in $B_{i,k}^*$ for at least one of the balls $B_{i,k}$ in $\mathfrak{B}_k$. Hence this ball is a subset of the complement of $\mathcal{E}_k^{\mathrm{sp}}$. Thus if D is any set of diameter $\le 2^k$ we get $\mathrm{card}(\mathcal{E}^{\mathrm{sp}} \cap D) \lesssim \lambda^p \,\mathrm{diam}(D)$. $\square$

We use the construction from Proposition 5.5 to build an exceptional set.

For each $k \in \mathbb{N}$ and for each ball $B \in \mathfrak{B}_k$, with center (y_B, r_B), we define the subset of $\mathbb{R}^d$

$$\mathcal{V}_B = \{x \in \mathbb{R}^d : \big||x - y_B| - r_B\big| \le 2(\mathrm{diam}(B^*) + 1)\}\,.$$

Observe that for $B \in \mathfrak{B}_k$,

$$\mathrm{supp}\Big(\sum_{(y,r)\in\mathcal{E}_k\cap B^*} c_{y,r} F_{y,r}\Big) \,\subset\, \mathcal{V}_B$$

and thus, if we define

$$\mathcal{V} \,=\, \bigcup_{k\in\mathbb{N}} \bigcup_{B\in\mathfrak{B}_k} \mathcal{V}_B \tag{5.18}$$

then

$$\mathrm{supp}\Big(\sum_k \sum_{(y,r)\notin\mathcal{E}_k^{\mathrm{sp}}} c_{y,r} F_{y,r}\Big) \,\subset\, \mathcal{V}\,. \tag{5.19}$$

Proposition 5.6. *The Lebesgue measure of $\mathcal{V}$ satisfies the estimate*

$$\mathrm{meas}(\mathcal{V}) \,\lesssim\, \lambda^{-p} \sum_k 2^{k(d-1)} \mathrm{card}(\mathcal{E}_k)\,.$$

Proof. Recall that for each $B \in \mathfrak{B}_k$ we have by property (i) in Proposition 5.5,

$$\mathrm{diam}(B) \le \lambda^{-p}\mathrm{card}(\mathcal{E}_k \cap B).$$

Thus

$$\begin{aligned}
\mathrm{meas}(\mathcal{V}) &\lesssim \sum_k \sum_{B\in\mathfrak{B}_k} 2^{k(d-1)}\mathrm{diam}(B^*)\\
&\lesssim \sum_k \sum_{B\in\mathfrak{B}_k} 2^{k(d-1)}\mathrm{diam}(B)\\
&\lesssim \sum_k \sum_{B\in\mathfrak{B}_k} 2^{k(d-1)}\lambda^{-p}\mathrm{card}(\mathcal{E}_k \cap B)\\
&\lesssim \lambda^{-p} \sum_k 2^{k(d-1)} \#\mathcal{E}_k\,.
\end{aligned}$$

$\square$

L^2-estimates. The purpose of this section is to prove

Proposition 5.7. *For $k = 1, 2, \dots$ let $\mathcal{E}_k^{\rm sp}$ be subsets of $\mathcal{Z}_{k,u}$ with the property that for any ball B of diameter $\le 2^k$ we have the sparseness assumption*

$$\operatorname{card}(\mathcal{E}_k^{\rm sp} \cap B) \lesssim \lambda^p (\operatorname{diam} B).$$

Then

$$\Big\| \sum_k \sum_{(y,r)\in\mathcal{E}_k^{\rm sp}} c_{y,r} F_{y,r} \Big\|_2^2 \lesssim \lambda^{\frac{2p}{d-1}} \log(2+\lambda) \sum_k 2^{k(d-1)} \# \mathcal{E}_k^{\rm sp}.$$

The proof is a combination of three lemmata. We begin with a straightforward L^2 estimate which does not rely on our weak orthogonality argument. Recall that always $\sup |c_{y,r}| \le 1$.

Lemma 5.8. *Let $1 \le L \le 2^k$ and I be a subinterval of $[2^k, 2^{k+1}]$ of length L. Let $\mathcal{E}_I$ be a subset of $\mathcal{Z}_{k,u}$ such that for all $(y,r) \in \mathcal{E}_I$ we have $r \in I$. Then*

$$\Big\| \sum_{(y,r)\in\mathcal{E}_I} c_{y,r} F_{y,r} \Big\|_2^2 \lesssim L 2^{k(d-1)} \# \mathcal{E}_I.$$

Proof. We have

$$\Big\| \sum_{y,r} c_{y,r} F_{y,r} \Big\|_2^2 = \Big\| \psi_0 * \widehat{\sigma}_r * \sum_{r\in I} \sum_{y:(y,r)\in\mathcal{E}_I} c_{y,r} \psi_0(\cdot - y) \Big\|_2^2$$

and, by the Cauchy-Schwarz inequality, this is estimated by a constant times

$$\begin{aligned}
& L \sum_{r\in I} \Big\| \psi_0 * \widehat{\sigma}_r * \sum_{y:(y,r)\in\mathcal{E}_I} c_{y,r} \psi_0(\cdot - y) \Big\|_2^2 \\
&\lesssim L \sum_{r\in I} r^{d-1} \Big\| \sum_{y:(y,r)\in\mathcal{E}_I} c_{y,r} \psi_0(\cdot - y) \Big\|_2^2 \\
&\lesssim L \sum_{r\in I} r^{d-1} \sum_{y:(y,r)\in\mathcal{E}_I} \|\psi_0(\cdot - y)\|_2^2 \\
&\lesssim L \sum_{r\in I} r^{d-1} \#\{y : (y,r) \in \mathcal{E}_I\}
\end{aligned}$$

where we have of course used that the y and r's are 1-separated. The last displayed quantity is bounded by $L 2^{k(d-1)} \#(\mathcal{E}_I)$. ☐

We now let $\lambda > 1$ and prove an L^2 estimate for the set $\mathcal{E}^{\mathrm{sp}}$ constructed in the previous section. We thus assume that every ball of radius $\le 2^k$ contains no more than $\lambda^p \operatorname{diam}(B)$ points in $\mathcal{E}_k^{\mathrm{sp}}$.

Lemma 5.9. *Let $1 \le L \le 2^k$ and let $\mathcal{I}$ be a collection of disjoint subintervals of $[2^{k-1}, 2^{k+2}]$ which are of length L. Let*

$$\mathcal{E}_{k,I} = \{(y,r) \in \mathcal{E}_k^{\mathrm{sp}} : r \in I\}.$$

Then

$$\Big\| \sum_{I \in \mathcal{I}} \sum_{(y,r)\in \mathcal{E}_{k,I}} c_{y,r} F_{y,r} \Big\|_2^2 \lesssim \big(L + \lambda^p L^{-\ell\frac{d-3}{2}}\big) 2^{k(d-1)} \# \mathcal{E}^{\mathrm{sp}}.$$

Proof. We may assume that the intervals are $10L$-separated (after splitting the collection $\mathcal{I}$ into eleven subcollections with this property). Let

$$G_{k,I} = \sum_{I\in\mathcal{I}} \sum_{(y,r)\in\mathcal{E}_{k,I}^{\mathrm{sp}}} c_{y,r} F_{y,r}.$$

Then

$$\Big\| \sum_I G_{k,I} \Big\|_2^2 \lesssim \sum_I \big\|G_{k,I}\big\|_2^2 + \sum_{I'\neq I} \big|\langle G_{k,I'}, G_{k,I}\rangle\big|.$$

By Lemma 5.8,

$$\sum_I \big\|G_{k,I}\big\|_2^2 \lesssim L \sum_I 2^{k(d-1)} \#(\mathcal{E}_{k,I}) \lesssim L 2^{k(d-1)} \#(\mathcal{E}_k^{\mathrm{sp}})\,. \tag{5.20}$$

We claim that

$$\sum_{I'\neq I} \big|\langle G_{k,I}, G_{k,I'}\rangle\big| \lesssim \lambda^p L^{-\frac{d-3}{2}} 2^{k(d-1)} \#\mathcal{E}_k. \tag{5.21}$$

Let $(y,r)\in\mathcal{E}_{k,I}$. Then by Lemma 5.2, part (i),

$$\begin{aligned}
&\sum_{I'\neq I} \sum_{(y',r')\in\mathcal{E}_{k,I'}} \big|\langle F_{y,r}, F_{y',r'}\rangle\big| \\
&\lesssim 2^{k(d-1)} \sum_{\substack{(y',r')\in\mathcal{E}_k^{\mathrm{sp}} \\ L\le |(y',r')-(y,r)|\le 2^{k+3}}} \big|(y',r')-(y,r)\big|^{-\frac{d-1}{2}} \\
&\lesssim 2^{k(d-1)} \sum_{2^\ell \ge L/2} 2^{-\ell\frac{d-1}{2}} (\lambda^p 2^\ell) \\
&\lesssim 2^{k(d-1)} \lambda^p L^{-\frac{d-3}{2}}\,.
\end{aligned}$$

where we have used that each ball of diameter 2^ℓ contains no more than $2^\ell\lambda^p$ points in $\mathcal{E}_k^{\rm sp}$. We also used $d>3$ to sum the geometric series. Now we sum in (y,r) and get

$$\begin{aligned}&\sum_{I'\neq I}\big|\langle G_{k,I},G_{k,I'}\rangle\big|\\&\le\sum_I\sum_{(y,r)\in\mathcal{E}_{k,I}}\sum_{I'\neq I}\sum_{(y',r')\in\mathcal{E}_{k,I'}}\big|\langle F_{y,r},F_{y',r'}\rangle\big|\\&\lesssim\sum_I\sum_{(y,r)\in\mathcal{E}_{k,I}}2^{k(d-1)}\lambda^pL^{-\ell\frac{d-3}{2}}\\&\lesssim\lambda^pL^{-\ell\frac{d-3}{2}}2^{k(d-1)}\#\mathcal{E}_k^{\rm sp}\,.\end{aligned}$$

This proves (5.21). The lemma follows if we combine (5.20) and (5.21). □

Corollary 5.10.

$$\Big\|\sum_{(y,r)\in\mathcal{E}_k^{\rm sp}}c_{y,r}F_{y,r}\Big\|_2^2\lesssim\lambda^{\frac{2p}{d-1}}2^{k(d-1)}\#\mathcal{E}_k^{\rm sp}.$$

Proof. If $2^k\le\lambda^{p\frac{2}{d-1}}$ this follows from Lemma 5.8. If $2^k>\lambda^{p\frac{2}{d-1}}$ this follows from Lemma 5.9 if we choose $L=\lambda^{\frac{2}{d-1}}$. □

Finally we show some almost orthogonality for the sums in which we allow to vary k.

Lemma 5.11.

$$\Big\|\sum_k\sum_{(y,r)\in\mathcal{E}_k^{\rm sp}}c_{y,r}F_{y,r}\Big\|_2^2\lesssim\lambda^{\frac{2p}{d-1}}\log(2+\lambda)\sum_k 2^{k(d-1)}\#\mathcal{E}^{\rm sp}.$$

Proof. Let $N(\lambda)$ be the smallest integer larger than $10\log_2(2+\lambda^p)$. Let

$$G_k=\sum_{(y,r)\in\mathcal{E}_k^{\rm sp}}c_{y,r}F_{y,r}.$$

We write

$$\Big\|\sum_k G_k\Big\|_2^2\le 2\Big\|\sum_{k\le N(\lambda)}G_k\Big\|_2^2+\Big\|\sum_{k>N(\lambda)}G_k\Big\|_2^2 \tag{5.22}$$

and estimate

$$\Big\| \sum_{k\le N(\lambda)} G_k \Big\|_2^2 \lesssim N(\lambda) \sum_{k\le N(\lambda)} \|G_k\|_2^2$$
$$\lesssim \lambda^{\frac{2p}{d-1}} \log(2+\lambda) \sum_{k\le N(\lambda)} 2^{k(d-1)} \#\mathcal{E}^{\rm sp}, \tag{5.23}$$

by Lemma 5.9.

Now, in order to estimate the L^2 norm of $\sum_{k>N(\lambda)} G_k$ we may assume the sum in k is taken over a 10-separed subsets of integers $\ge N(\lambda)$ (just split the original sum in eleven different sums with this property). We then have

$$\Big\| \sum_{k>N(\lambda)} G_k \Big\|_2^2 \le \sum_{k>N(\lambda)} \|G_k\|_2^2 + 2 \sum_{k>k'>N(\lambda)} \big|\langle G_k, G_{k'}\rangle\big| \,. \tag{5.24}$$

Again, by Lemma 5.9,

$$\sum_{k>N(\lambda)} \|G_k\|_2^2 \lesssim \lambda^{\frac{2p}{d-1}} \sum_{k>N(\lambda)} 2^{k(d-1)} \#\mathcal{E}^{\rm sp} \,. \tag{5.25}$$

To estimate the second term in (5.24) we fix

$$(y,r) \in \mathcal{E}^{\rm sp}_k.$$

Let $k' < k-10$ and let $(y',r') \in \mathcal{E}^{\rm sp}_{k'}$. Then $F_{y',r'}$ is supported on a ball of radius $r'+1$ centered at y' and $F_{y,r}$ is supported on the 1 neighborhood of the sphere of radius r centered at y. This means $\langle F_{y,r}, F_{y',r'}\rangle$ can be different from zero only if

$$r - 2^{k'+3} \le |y-y'| \le r + 2^{k'+3}.$$

The set of $(y',r') \in \mathcal{Z}_u$ satisfying $2^{k'} \le r \le 2^{k'+1}$ and satisfying the displayed inequality can be covered with $O(2^{(k-k')(d-1)})$ balls of radius $2^{k'}$ and each of them contains no more than $O(\lambda^p 2^{k'})$ points of $\mathcal{E}^{\rm sp}_{k'}$. For those points the distance of (y,r) and (y',r') is $\gtrsim 2^k$ and therefore, by Lemma 5.13,

$$|\langle F_{y,r}, F_{y',r'}\rangle| \lesssim (rr')^{\frac{d-1}{2}} 2^{-k\frac{d-1}{2}} \lesssim 2^{k'\frac{d-1}{2}} \,.$$

Then also

$$\sum_{(y',r')\in \mathcal{E}^{\rm sp}_{k'}} |\langle F_{y,r}, F_{y',r'}\rangle| \lesssim 2^{(k-k')(d-1)} (\lambda^p 2^{k'}) 2^{k'\frac{d-1}{2}} \lesssim \lambda^p 2^{k(d-1)} 2^{-k'\frac{d-3}{2}} \,.$$

Hence

$$\begin{aligned}&\sum_{N(\lambda)<2^{k'}<2^{k-10}}\ \sum_{(y',r')\in\mathcal{E}^{\rm sp}_{k'}} |\langle F_{y,r}, F_{y',r'}\rangle|\\
&\lesssim \lambda^p 2^{k(d-1)} \sum_{N(\lambda)<2^{k'}<2^{k-10}} 2^{-k'\frac{d-3}{2}}\\
&\lesssim \lambda^p 2^{-N(\lambda)\frac{d-3}{2}} 2^{k(d-1)} \lesssim 2^{k(d-1)}\end{aligned}$$

since $d>3$.

Finally we sum over all $(y,r)\in\mathcal{E}^{\rm sp}_k$ and get from the last display

$$\begin{aligned}&\sum_{k>k'>N(\lambda)} \big|\langle G_k, G_{k'}\rangle\big|\\
&\lesssim \sum_{k>N(\lambda)}\ \sum_{(y,r)\in\mathcal{E}^{\rm sp}_k}\ \sum_{N(\lambda)<2^{k'}<2^{k-10}}\ \sum_{(y',r')\in\mathcal{E}^{\rm sp}_{k'}} |\langle F_{y,r}, F_{y',r'}\rangle|\\
&\lesssim \sum_{k>N(\lambda)}\ \sum_{(y,r)\in\mathcal{E}^{\rm sp}_k} 2^{k(d-1)} \lesssim 2^{k(d-1)}\,\#\mathcal{E}^{\rm sp}_k. \end{aligned}\tag{5.26}$$

If we combine the last estimate with (5.25) we obtain

$$\Big\|\sum_{k>N(\lambda)} G_k\Big\|_2^2 \lesssim \lambda^p 2^{k(d-1)}\,\#\mathcal{E}^{\rm sp}_k.$$

and this, together with (5.23), proves the lemma. □

Conclusion. We prove the restricted weak type inequality (5.17), for $\lambda>1$. Let $\mathcal{V}$ be as in (5.18). Using (5.19) we get

$$\begin{aligned}&\operatorname{meas}\big(\{x\in\mathbb{R}^d: \Big|\sum_k \sum_{(y,r)\in\mathcal{E}_k} c_{y,r}F_{y,r}\Big|>\lambda\}\big)\\
&\lesssim \operatorname{meas}(\mathcal{V}) + \operatorname{meas}\big(\{x\in\mathbb{R}^d: \Big|\sum_k \sum_{(y,r)\in\mathcal{E}^{\rm sp}_k} c_{y,r}F_{y,r}\Big|>\lambda\}\big)\\
&\lesssim \operatorname{meas}(\mathcal{V}) + \lambda^{-2}\Big\|\sum_k \sum_{(y,r)\in\mathcal{E}^{\rm sp}_k} c_{y,r}F_{y,r}\Big\|_2^2,\end{aligned}$$

by Tshebyshev's inequality.

By Proposition 5.6 and Proposition 5.7 we get

$$\begin{aligned} &\mathrm{meas}\big(\{x\in\mathbb{R}^d: \Big|\sum_k \sum_{(y,r)\in\mathcal{E}_k} c_{y,r}F_{y,r}\Big| > \lambda\}\big) \\ &\lesssim \big(\lambda^{-p}+\lambda^{\frac{2p}{d-1}-2}\log(2+\lambda^p)\big)\sum_k 2^{k(d-1)}\lambda^{-p}\#\mathcal{E}_k\,, \end{aligned}$$

and the proof is concluded by observing that for $\lambda>1$

$$\lambda^{\frac{2p}{d-1}-2}\log(2+\lambda^p)\lesssim\lambda^{-p} \quad \text{if} \quad p<\frac{2(d-1)}{d+1}\,.$$

6. Characterization of radial Fourier multipliers and further results

One can generalize the theorem in the previous section to obtain a characterization of all L^p bounded operators which commute with rotations and translations, in the range $1<p<\frac{2(d-1)}{d+1}$. The following is proved in [24].

Theorem 6.1. *Let $m=h(|\cdot|)$ be a bounded radial function on $\mathbb{R}^d$ and define the convolution operator T_h on $\mathbb{R}^d$ by*

$$\widehat{T_h f}(\xi)=h(|\xi|)\widehat{f}(\xi). \tag{6.1}$$

Let $1<p_1<\frac{2d}{d+1}$ and assume that (5.7) holds for p_1. Let $1<p<p_1$, and let η be any nontrivial Schwartz function on $\mathbb{R}^d$ and let ϕ be any nonzero C^∞_c function compactly supported on $(0,\infty)$. Then the following statements are equivalent.

(i) T_h is bounded on L^p.

(ii) T_h maps L^p_{rad} to L^p_{rad}.

(iii) $\sup_{t>0} t^{d/p}\big\|T_h[\eta(t\cdot)]\big\|_p<\infty$

(iv) The functions $\kappa_t=\mathcal{F}^{-1}_{\mathbb{R}}[\phi h(t\cdot)]$ belong to $L^p((1+|r|)^{(d-1)(1-p/2)}dr)$ with norm uniformly bounded independently of t.

The proof is somewhat technical and uses atomic decompositions of L^p spaces. By the results of the previous sections we have the following corollary (proved first in [23]).

Corollary 6.2. *Let $d\ge 4$ and $1<p<\frac{2(d-1)}{d+1}$. Then statements (i) - (iv) are equivalent.*

It would be very interesting to get a sharp inequality in two and three dimensions, for some nontrivial range of p.

It is easy to see (by Hölder's inequality on dyadic intervals and Plancherel's theorem) that

$$\Big(\int \big|\kappa(r)(1+|r|)^{(d-1)(1-p/2)}dr\Big)^{1/p} \lesssim \|\widehat{\kappa}\|_{B^2_{d/p-d/2,p}}\,. \tag{6.2}$$

Thus Conjecture II, for $d \ge 4$ in the range $p < \frac{2(d-1)}{d+1}$ follows from Conjecture III . However one can modify the proofs of Theorems 5.1 and 6.1 to get Conjecture II and its global analogue in the range of the Tomas restriction theorem for the sphere, see [30].

Theorem 6.3. *For $d \ge 2$, $1 < p < \frac{2(d+1)}{d+3}$,*

$$\big\|\mathcal{F}^{-1}[h(|\cdot|)\widehat{f}]\big\|_p \lesssim \sup_{t>0} \|\varphi h(t\cdot)\|_{B^2_{d(\frac1p-\frac12),p}} \|f\|_p\,.$$

Finally, the upper bounds in Corollary 6.2 and Theorem 2.2 can be interpolated (using Calderón's $[\cdot,\cdot]^\theta$-method, see [3]) to get the following M_p^s theorem, for s between p and 2.

Theorem 6.4. *Let $d \ge 4$, and let $m = h(|\cdot|)$ be a bounded radial function on $\mathbb{R}^d$ and let T_h be as in* (6.1). *Let either*

(a) $1 \le p < \frac{2(d-1)}{d+1}$ and $p \le s \le 2$,

or

(b) $\frac{2(d-1)}{d+1} \le p < \frac{2(d+1)}{d+3}$ and $\frac12 < \frac1s < \frac{d+1}{2}(\frac1p - \frac12)$.

Then the following are equivalent:

(i) T_h maps L^p to L^s.

(ii) T_h maps L^p_{rad} to L^s_{rad}.

(iii) $\sup_{t>0} t^{d/p}\big\|T_h[\eta(t\cdot)]\big\|_s < \infty$

(iv) Let $\kappa_t = \mathcal{F}_{\mathbb{R}}^{-1}[\phi h(t\cdot)]$; then the function $t^{d/p-d/s}\kappa_t$ belong to $L^s((1+|r|)^{(d-1)(1-s/2)}dr)$ with norm uniformly bounded independently of t.

Similarly, the M_p^s analogue of Theorem

Theorem 6.5. *For $d \ge 2$, $1 < p < \frac{2(d+1)}{d+3}$, $p \le s \le 2$,*

$$\big\|\mathcal{F}^{-1}[h(|\cdot|)\widehat{f}]\big\|_s \lesssim \sup_{t>0} t^{d(1/p-1/s)} \|\varphi h(t\cdot)\|_{B^2_{d(\frac1s-\frac12),s}} \|f\|_p\,.$$

A Hardy space bound. The following Hardy space result is more elementary than Theorem 6.4. We let φ be a C_0^∞ function supported in $(1/2, 2)$ such that $\sum_{k\in\mathbb{Z}} \varphi(2^k\cdot) = 1$.

Theorem 6.6. *Let $d \geq 2$. For $f \in \mathcal{S}(\mathbb{R}^d)$ define T by $\widehat{Tf}(\xi) = h(|\xi|)\widehat{f}(\xi)$ and T_k by $\widehat{T_k f}(\xi) = \varphi(2^{-k}|\xi|)\widehat{Tf}(\xi)$. Then*

$$\|T\|_{H^1 \to L^1} \approx \sup_k \|T_k\|_{L^1 \to L^1}.$$

Proof. Clearly if $T : H^1 \to L^1$ then $T_k : L^1 \to L^1$ with uniform bounds.

Vice versa let K_k be the radial kernel with $\widehat{K}_k = \varphi(|\xi|)h(2^k|\xi|)$. Then, as before we can write $K_k = \int_0^\infty \mathcal{K}_k(r)\psi * \sigma_r dr$ where $\psi = \psi_\circ * \psi_\circ * \psi_\circ * \psi_\circ$, $\psi_\circ$ is supported in a ball of radius $1/4$ and all moments of order $\leq 5d$ of $\psi_\circ$ vanish. Let

$$B = \sup_{k \in \mathbb{Z}} \int_0^\infty |\mathcal{K}_k(r)|(1+r)^{d-1} dr \approx \sup_k \|T_k\|_{L^1 \to L^1} .$$

We use the atomic decomposition of H^1 (see e.g. [47]). By translation invariance and by dilation invariance of the hypothesis we need to prove

(6.3)
$$\Big\| \sum_{k \in \mathbb{Z}} \int_0^\infty \mathcal{K}_k(r) 2^{kd} [\psi * \sigma_r](2^k \cdot) * a\, dr \Big\|_1 \lesssim \sup_{k \in \mathbb{Z}} \int_0^\infty |\mathcal{K}_k(r)|(1+r)^{d-1} dr$$

where $\mathrm{supp}(a) \subset \{y : |y| \leq 1\}$, $\|a\|_\infty \leq 1$ and $\int a(x)dx = 0$.

For $k < 0$ we use the cancellation condition. We have

$$\|2^{kd}[\psi * \sigma_r](2^k \cdot) * a\|_1 \lesssim (1+r)^{d-1} \|2^{kd}\psi(2^k \cdot) * a\|_1$$

and since $\|2^{kd}\psi(2^k \cdot) * a\|_1 = O(2^k)$ we see using Minkowski's inequality

$$\Big\| \sum_{k \leq 0} \int_0^\infty \mathcal{K}_k(r) 2^{kd} [\psi * \sigma_r](2^k \cdot) * a\, dr \Big\|_1 \lesssim B \sum_{k \leq 0} 2^k \lesssim B.$$

Next we consider the corresponding sum for $k > 0$. Note that the expression

$$\int_0^{2^{k+10}} \mathcal{K}_k(r) 2^{kd} [\psi * \sigma_r](2^k \cdot) * a\, dr$$

is supported in $\{x : |x| \leq C\}$ and for every $n \geq 2^{10}$ the expression

$$\int_{2^k n}^{2^k(n+1)} \mathcal{K}_k(r) 2^{kd} [\psi * \sigma_r](2^k \cdot) * a\, dr$$

is supported in an annulus centered at 0 with radius $\approx n$ and width ≈ 1.

We set $\psi_{\circ,k} = 2^{kd}\psi_\circ(2^k\cdot)$. Apply the Cauchy-Schwarz inequality on $\{|x| \le C\}$ and (5.3) to get

$$\begin{aligned}
&\Big\| \sum_{k\ge 0} \int_0^{2^{k+10}} \mathcal{K}_k(r) 2^{kd}[\psi * \sigma_r](2^k\cdot) * a\, dr \Big\|_1 \\
&\lesssim \Big\| \sum_{k\ge 0} \psi_{\circ,k} * \int_0^{2^{k+10}} \mathcal{K}_k(r) 2^{kd}[\psi_\circ * \psi_\circ * \sigma_r](2^k\cdot) * \psi_{\circ,k} * a\, dr \Big\|_2 \\
&\lesssim \Big(\sum_{k\ge 0} \Big\| \int_0^{2^{k+10}} \mathcal{K}_k(r) 2^{kd}[\psi_\circ * \psi_\circ * \sigma_r](2^k\cdot) * \psi_{\circ,k} * a\, dr \Big\|_2^2 \Big)^{1/2} \\
&\lesssim \Big(\sum_{k\ge 0} \Big[\int_0^{2^{k+10}} |\mathcal{K}_k(r)| \|\widehat{\psi_\circ * \sigma_r}\|_\infty \|\psi_{\circ,k} * a\|_2\, dr \Big]^2 \Big)^{1/2} \Big\|_2 \\
&\lesssim B (\sum_k \|\psi_{\circ,k} * a\|_2^2)^{1/2} \lesssim B\,.
\end{aligned}$$

Finally, applying Cauchy-Schwarz on $\{x : ||x| - n| \le C\}$, and using (5.3) again yields

$$\begin{aligned}
&\Big\| \sum_{k\ge 0} \sum_{n\ge 2^{10}} \int_{2^{k+n}}^{2^{k+n+1}} \mathcal{K}_k(r) 2^{kd}[\psi * \sigma_r](2^k\cdot) * a\, dr \Big\|_1 \\
&\lesssim \sum_{k\ge 0} \sum_{n\ge 2^{10}} n^{\frac{d-1}{2}} \Big\| \int_{2^k n}^{2^k(n+1)} \mathcal{K}_k(r) 2^{kd}[\psi * \sigma_r](2^k\cdot) * a\, dr \Big\|_2 \\
&\lesssim \sum_{k\ge 0} \sum_{n\ge 2^{10}} n^{\frac{d-1}{2}} \int_{2^k n}^{2^k(n+1)} |\mathcal{K}_k(r)| r^{\frac{d-1}{2}}\, dr\, \|a\|_2 \\
&\lesssim \sum_{k\ge 0} 2^{-k\frac{d-1}{2}} \sum_{n\ge 2^{10}} \int_{2^k n}^{2^k(n+1)} |\mathcal{K}_k(r)| r^{d-1} dr\, \|a\|_2 \lesssim B\,.
\end{aligned}$$

□

Remark. There is no equivalent of Theorem 6.6 in the case $d = 1$. Consider the even multipliers on the real line

$$m_N(\xi) = \sum_{k=N}^{2N} e^{i2^k|\xi|}\chi(|\xi| - 2^k)$$

where $\chi \in C_0^\infty(\mathbb{R})$. Then the dyadic pieces represent multipliers with L^1 norm uniform in k, but the $H^1(\mathbb{R}) \to L^1(\mathbb{R})$ operator norm is $\approx \sqrt{N}$

for large N; moreover for $1 < p \le 2$, $\|m_N\|_{M_p} \approx N^{1/p-1/2}$. See [34] and [50]. This example is in some sense optimal, see [10], [43].

An L^p space time estimate for the wave equation. For the proof of our L^p multiplier theorems we need the main inequality (5.7) only for tensor product functions of the form $g(y,r) = f(y)\kappa(r)$. However using duality, the more general inequality can be used to prove sharp L^q-Sobolev space time estimates for the wave equation (see [23], [24]).

Theorem 6.7. *For $d \ge 4$, $q > \frac{2(d-1)}{d-3}$,*

$$\Big(\int_{-1}^{1} \big\|e^{it\sqrt{-\Delta}} f\big\|_q^q dt\Big)^{1/q} \lesssim \|f\|_{L^q_\alpha}, \quad \alpha = d(\frac{1}{2} - \frac{1}{q}) - \frac{1}{2}\,. \tag{6.4}$$

This can be compared with the fixed-time bound (Peral [40])

$$\big\|e^{it\sqrt{-\Delta}} f\big\|_q^q \lesssim \|f\|_{L^q_\beta}^q, \qquad \beta = (d-1)(\frac{1}{2} - \frac{1}{q})\,.$$

Sogge's "local smoothing" conjecture [45] says that for $q > \frac{2d}{d-1}$ the inequality (6.4) should be true with $\alpha > \beta - 1/q$. That is the t integration gains almost $1/q$ derivatives over the fixed time estimate. Hence Theorem 6.7 realizes an endpoint version of Sogge's conjecture in a restricted q-range. For a variable coefficient analogue and other extensions see [31]. The theorem improves, in high dimensions, the p range in proofs based on Wolff's inequality for plate decompositions of cone multipliers ([59], [27]). In low dimensions the Wolff method gives better (although currently no endpoint) results. Finally, Lee and Vargas proved more recently that the wave operator maps $L^q_\varepsilon(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2 \times [-1,1])$, for any $\varepsilon > 0$ and $2 < q \le 3$. Their method is different and relies on multilinear adjoint restriction estimates by Bennett, Carbery and Tao [2] and the method by Bourgain and Guth [7].

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