LECTURE NOTES: ON EXISTENCE THEORY FOR GENERAL NONLINEAR ELLIPTIC AND PARABOLIC EQUATIONS WITH BAD DATA

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1. Introduction

In many physical applications, the diffusion and/or the dissipation play the crucial leading role. Such processes are then in the first approximation usually modeled by the Laplace (the steady case) or by the heat (the evolutionary case) equations and they could provide an useful insight into the physical phenomena we have in

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mind. Unfortunately, as mentioned above, these models are just the first approximation[1] of the reality. Therefore, in many applications it is of a real importance to include also other natural phenomena that can not be covered by linear Laplace or heat equation. We focus on three different aspects of such a generalization, namely we want to include into the theory the following phenomena:

(P1) to consider the general fully nonlinear elliptic-like operator having almost no structure;
(P2) to include the transport term into the equation;
(P3) to include as bad data as possible.

Let us briefly discuss the requirements (P1)–(P3). Concerning (P1), frequently the diffusion is described by a linear second order operator. This is a very good approximation in case that the fluxes (gradients of the unknown) are small or at least not large. However, if we allow in the model the presence of large gradients the liner operator is not a sufficient approximation of the diffusion. Moreover, not only nonlinearity may appear in the elliptic part but also lot of singular phenomena can be presented, as for example certain thresholds making the elliptic part discontinuous with respect to the unknown, dependence on other variables/unknowns that are not smooth, etc. see below. To cover such wild behavior we employ the concept of the maximal monotone graph, which seems to be the most proper tool for dealing with such discontinuities. Concerning (P2), in many situation, not only diffusion takes place but more important transport effects are observed and therefore can not be neglected. In particular in continuum mechanics the transport term (or the convection) usually dominates the problem and is driven by a velocity field that has very poor regularity/integrability. Finally and most importantly, (P3) is the leading difficulty in the analysis of the problem. Not only it is challenging from the mathematical point of view but more importantly it is also the situation we must deal with in many interesting application. Typically, the source terms in the equations may be a measure supported on a compact subset representing either the volume or the surface measure. In addition, coming again back to the continuum mechanics, the right hand side of the heat equation represents the source of dissipation, which is a priori exactly only in the space L^1 or even worse in the space of measures. From all these reasons one must focus on all (P1)–(P3) together and not only develop a theory for particular cases.

1.1. The problem formulation.

We look for a solution $(u, \boldsymbol{q})$ to the following scalar non-linear parabolic equation[2]

$$u_{,t} + \operatorname{div} \boldsymbol{g}(\cdot, u) - \operatorname{div} \boldsymbol{q} = f \quad \text{in } Q := (0,T) \times \Omega, \tag{1.1}$$

where $T > 0$ is the length of time interval and Ω is a bounded Lipschitz domain in $\mathbb{R}^d$ with $d \geq 2$, or its steady variant

$$\operatorname{div} \boldsymbol{g}(\cdot, u) - \operatorname{div} \boldsymbol{q} = f \quad \text{in } \Omega. \tag{1.2}$$

The equation (1.1) is completed by the initial condition

$$u(0,x) = u_0(x) \quad (x \in \Omega) \tag{1.3}$$

[1] In fact, any model is an approximation of the reality, and our goal is to fit the parameter of the model such that it describes the real experiments in the range of the experimental data/setting as precisely as possible

[2] Other names for (1.1) and (1.2) are *non-linear heat equation with convection* or *non-linear convection-diffusion equation*.

and for (1.1)–(1.2) we consider the homogeneous Dirichlet boundary conditions[3], i.e.,

$$u(t,x) = 0 \text{ for } (t,x) \in (0,T) \times \partial\Omega \qquad \text{or} \qquad u(x) = 0 \text{ for } x \in \partial\Omega. \tag{1.4}$$

To avoid any ambiguity in formulating further assumptions we employ the notation $\mathcal{O} := Q$ and $z := (t,x)$ if we consider (1.1), i.e., the parabolic case, and $\mathcal{O} := \Omega$ with $z := x$ if we consider (1.2), the elliptic case. For sake of clarity, we also recall the meaning of differential operators appearing in (1.1)–(1.2)

$$u_{,t} := \frac{\partial u}{\partial t}, \qquad \operatorname{div} \boldsymbol{v} := \sum_{i=1}^{d} \frac{\partial v_i}{\partial x_i}$$

for all $u : \mathcal{O} \to \mathbb{R}$ and all $\boldsymbol{v} = (v_1, \ldots, v_d) : \mathcal{O} \to \mathbb{R}^d$. Further, we assume that $\boldsymbol{g} : \mathcal{O} \times \mathbb{R} \to \mathbb{R}^d$ and $f : \mathcal{O} \to \mathbb{R}$ are given data. To complete the problem we must describe the relationship between u and $\boldsymbol{q}$. Thus, we assume that there is a given

$$F : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$$

and we require that for almost all $z \in \mathcal{O}$ there holds

$$F(z, u(z), \nabla_x u(z), \boldsymbol{q}(z)) = 0. \tag{1.5}$$

1.2. **Assumptions on data.** Here, we specify the assumptions on data. In what follows we assume that C_1 and C_2 are given positive constants. First, we specify the assumptions on F appearing in (1.5). For any $z \in \mathcal{O}$ and $u \in \mathbb{R}$ we identify the null points of F with the graph $\mathcal{A}(z,u) \subset \mathbb{R}^d \times \mathbb{R}^d$, i.e., we say that for all $(\boldsymbol{w}, \boldsymbol{u}) \in \mathbb{R}^d \times \mathbb{R}^d$

$$(\boldsymbol{w}, \boldsymbol{u}) \in \mathcal{A}(z,u) \iff F(z, u, \boldsymbol{w}, \boldsymbol{u}) = 0 \tag{1.6}$$

and we assume that there exist $q \in (1,\infty)$ such that for almost all $z \in \mathcal{O}$ and all $u \in \mathbb{R}$, the graph $\mathcal{A}(z,u)$ is the maximal monotone q-graph that means

(A1) the graph contains the origin, i.e.,
$$(\boldsymbol{0}, \boldsymbol{0}) \in \mathcal{A}(z,u).$$
(A2) the graph is monotone, i.e., for all $(\boldsymbol{w}_1, \boldsymbol{u}_1), (\boldsymbol{w}_2, \boldsymbol{u}_2) \in \mathcal{A}(z,u)$ there holds
$$(\boldsymbol{w}_1 - \boldsymbol{w}_2) \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2) \geq 0.$$
(A3) the graph is maximal, i.e., if for some $(\boldsymbol{w}, \boldsymbol{u}) \in \mathbb{R}^d \times \mathbb{R}^d$ and all $(\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{u}}) \in \mathcal{A}(z,u)$ the following holds
$$(\boldsymbol{w} - \tilde{\boldsymbol{w}}) \cdot (\boldsymbol{u} - \tilde{\boldsymbol{u}}) \geq 0,$$
then $(\boldsymbol{w}, \boldsymbol{u}) \in \mathcal{A}(z,u)$.

(A4) the graph is q-coercive, i.e., for all $(\boldsymbol{w}, \boldsymbol{u}) \in \mathcal{A}(z,u)$ there holds
$$\boldsymbol{w} \cdot \boldsymbol{u} \geq C_1(|\boldsymbol{w}|^q + |\boldsymbol{u}|^{q'}) - C_2,$$
where $q' := \frac{q}{q-1}$.

[3] The use of a homegeneous Dirichlet data is not essential for the analysis. One could also use the inhomogeneous conditions as well as the Neumann or the Newton type of boundary conditions.

(A5) there exists a measurable selection, i.e., there exists $\boldsymbol{q}^*:\mathcal{O}\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^d$ such that for almost all $z\in\mathcal{O}$, all $u\in\mathbb{R}$ and all $\boldsymbol{w}\in\mathbb{R}^d$ there holds

$$(\boldsymbol{w},\boldsymbol{q}^*(z,u,\boldsymbol{w}))\in\mathcal{A}(z,u).$$

Moreover, we require that $\boldsymbol{q}^*$ is measurable with respect to z, continuous with respect to u and Borel measurable with respect to $\boldsymbol{w}$.

In addition, having the existence of selection, we can equivalently replace (A3) by the following requirement

(A6) if for some $(\boldsymbol{w},\boldsymbol{u})\in\mathbb{R}^d\times\mathbb{R}^d$ and all $\tilde{\boldsymbol{w}}\in\mathbb{R}^d$ the following holds

$$(\boldsymbol{w}-\tilde{\boldsymbol{w}})\cdot(\boldsymbol{u}-\boldsymbol{q}^*(z,u,\tilde{\boldsymbol{w}}))\ge 0,$$

then $(\boldsymbol{w},\boldsymbol{u})\in\mathcal{A}(z,u)$.

Next, for the term on the right hand side, we assume

$$f\in\mathcal{M}(\mathcal{O}),\qquad\text{or}\qquad f\in L^1(\mathcal{O})\tag{1.7}$$

and similarly for the initial data, we consider

$$u_0\in\mathcal{M}(\Omega),\qquad\text{or}\qquad u_0\in L^1(\Omega).\tag{1.8}$$

Note that L^1 stands for the standard Lebesgue space, while $\mathcal{M}$ denotes the space of measures. Finally, for $\boldsymbol{g}$ we consider that it is of the form

$$\boldsymbol{g}(z,u):=\boldsymbol{v}_{\mathrm{div}}(z)u+\boldsymbol{v}(z)g(u),\tag{1.9}$$

where

$$\boldsymbol{v}\in L^r(\mathcal{O}),\qquad \boldsymbol{v}_{\mathrm{div}}\in L^s(\mathcal{O})\qquad\text{and}\qquad \operatorname{div}\boldsymbol{v}_{\mathrm{div}}\equiv 0\text{ in }\mathcal{O},\tag{1.10}$$

for some given $r,s\in[1,\infty]$ and $g:\mathbb{R}\to\mathbb{R}$ is a continuous function fulfilling the growth estimate

$$|g(u)|\le C(1+|u|)^{(q-1)p}\tag{1.11}$$

for some $p\in[0,\infty)$.

1.3. **Notation.** We briefly recall the notation for standard Lebesgue and Sobolev spaces. For $p\in[1,\infty]$ and $k\in\mathbb{R}_+$, we use the symbols $L^p(\Omega)$ and $W^{k,p}(\Omega)$ for Lebesgue and Sobolev spaces. We also define $W^{k,p}_0(\Omega):=\{\varphi;\ \varphi\in W^{k,p}(\Omega),\ \varphi|_{\partial\Omega}=0\}$ and we denote its dual $(W^{k,p}_0(\Omega))^*$ by $W^{-k,p'}(\Omega)$. For a Banach space X, we introduce the standard Bochner space $(L^p(0,T;X),\|\cdot\|_{p;X})$. Further, $\mathcal{M}(\mathcal{O})$ denotes the spaces of sign measures on the set $\mathcal{O}$. We also write $\int_A a\cdot b=:(a,b)_A$ whenever $a\in L^q(A)$ and $b\in L^{q'}(A)$, and we use $\langle\cdot,\cdot\rangle_A$ to denote the duality pairing between various spaces $X(A)$ and $X^*(A)$ whenever it will be clear from the context which X is taken into account. Finally, to simplify the presentation, we introduce a variant of the grand L^p space and we denote $L^{p)}(\mathcal{O}):=\{u;\ u\in L^r(\mathcal{O})\text{ for all }r\in[1,p)\}$ and similarly for the Sobolev spaces. Moreover, we do not specify the explosion rate and simply write $\|u\|_{p)}\le C$ whenever $\|u\|_r\le C(r)$ for all $r\in[1,p)$, where we admit the possibility that $C(r)\to\infty$ as $r\to p_-$.

In what follows, we will also use various truncation function. Hence, for k, $\delta \in (0,\infty)$ we define T_k and its δ-mollification $T_{k,\delta}$ as

$$T_k(z) := \begin{cases} z & \text{if } |z| \le k, \\ \operatorname{sign}(z)k & \text{if } |z| > k, \end{cases} \tag{1.12}$$

$$T_{k,\delta}(z) := \begin{cases} z & \text{if } |z| \le k, \\ \operatorname{sign}(z)(k+\delta/2) & \text{if } |z| \ge k+\delta, \end{cases} \tag{1.13}$$

whereas $T_{k,\delta}$ is defined on $(k, k+\delta)$ in such a way that $T_{k,\delta} \in \mathcal{C}^2(\mathbb{R})$, $0 \le T'_{k,\delta} \le 1$ uniformly with respect to δ for all $z \in \mathbb{R}$, and $T_{k,\delta}$ is concave on $\mathbb{R}_+$ and convex on $\mathbb{R}_-$. The functions Θ_k and $\Theta_{k,\delta}$ denote the primitive functions to T_k and $T_{k,\delta}$ respectively, that means

$$\Theta_k(s) := \int_0^s T_k(t)\, dt, \qquad \Theta_{k,\delta}(s) := \int_0^s T_{k,\delta}(t)\, dt. \tag{1.14}$$

We also define the function $\chi_k := \chi_k(u)$ as

$$\chi_k(u) := \begin{cases} 1 & \text{if } |u| \le k, \\ 0 & \text{if } |u| > k. \end{cases}$$

Frequently, we also use the notation $T'_k(u) := \chi_k(u)$ although it is not valid if $|u| = k$.

2. Uniform a priori estimates

The aim of this section is to provide uniform a priori estimates for sufficiently smooth solutions to (1.1) and (1.2) in terms of data under the assumptions introduced in the previous section. These kind of estimates was first observed by Boccardo and Murat [1992], Boccardo et al. [1997], but we present here a slightly different (but in fact completely equivalent) hopefully easier method. Thus, in what follows the constant C can depend on the Ω, q, C_1, C_2 and can vary from line to line.

2.1. **The key estimates.**

In this subsection we focus on the standard a priori estimate for heat-like or parabolic-like estimates with the right hand side in L^1. Thus, the purpose of this subsection is to derive the estimates depending only $\|f\|_1$, $\|u_0\|_1$ but still possibly on $\boldsymbol{v}$ and $g(u)$. Note that these estimates will not depend on $\boldsymbol{v}_{\mathrm{div}}$. We start with the following elementary observation.

Lemma 2.1. *Let u be a "sufficiently" smooth[4] solution to* (1.1) *or* (1.2) *then the following inequalities hold for all $k \in \mathbb{R}_+$ and all $t \in [0,T]$*

$$\begin{aligned} \|\Theta_k(u(t))\|_1 + \int_0^t (\boldsymbol{q}, \nabla u \chi_k)_\Omega \, d\tau &\le C \int_0^t \int_\Omega |f||u|\chi_k + |\boldsymbol{v}|^{q'} |g(u)|^{q'} \chi_k \, dx \, d\tau \\ &+ Ck \int_0^t \int_\Omega |f|(1-\chi_k)\, dx\, d\tau + C\left(1 + \|\Theta_k(u_0)\|_1\right) \end{aligned} \tag{2.1}$$

and in the steady case

$$(\boldsymbol{q}, \nabla u \chi_k)_\Omega \le C \int_\Omega |f||u|\chi_k + |\boldsymbol{v}|^{q'} |g(u)|^{q'} \chi_k \, dx + Ck \int_\Omega |f|(1-\chi_k)\, dx. \tag{2.2}$$

[4]Here sufficiently smooth means, we can use u as a test function.

Proof. We will prove only (2.1). Multiplying (1.1) by $T_k(u)$, integrating over Ω and using integration by parts, we observe that

$$(u_t, T_k(u))_\Omega + (\boldsymbol{q}, \nabla T_k(u))_\Omega = (f, T_k(u))_\Omega - (\operatorname{div}(\boldsymbol{v}_{\mathrm{div}} u + \boldsymbol{v} g(u)), T_k(u))_\Omega. \tag{2.3}$$

Next, using (1.14), it is evident that the first term can be rewritten as

$$(u_t, T_k(u))_\Omega = \frac{d}{dt}\|\Theta_k(u(t))\|_1.$$

Similarly, since $\operatorname{div} \boldsymbol{v}_{\mathrm{div}} = 0$ and u has zero trace we obtain for the term on the right hand side

$$\begin{aligned}(\operatorname{div}(\boldsymbol{v}_{\mathrm{div}} u), T_k(u))_\Omega &= (\boldsymbol{v}_{\mathrm{div}}, \nabla u T_k(u))_\Omega = (\boldsymbol{v}_{\mathrm{div}}, \nabla \Theta_k(u))_\Omega \\ &= -(\operatorname{div} \boldsymbol{v}_{\mathrm{div}}, \Theta_k(u))_\Omega = 0,\end{aligned}$$

where we also used the integration by parts. For the last term on the right hand side we use the integration by parts and the Young inequality to obtain

$$\begin{aligned}-(\operatorname{div}(\boldsymbol{v} g(u)), T_k(u))_\Omega &= (\boldsymbol{v} g(u), \nabla T_k(u))_\Omega \le C\|\boldsymbol{v} g(u)\chi_k\|_{q'}^{q'} + \frac{C_1}{2}\|\nabla u \chi_k\|_q^q \\ &\overset{(A4)}{\le} C\|\boldsymbol{v} g(u)\chi_k\|_{q'}^{q'} + \frac{1}{2}(\boldsymbol{q}, \nabla u \chi_k)_\Omega + \frac{C_2}{2},\end{aligned}$$

where we used the identity

$$\nabla T_k(u) = \nabla u \chi_k.$$

Upon inserting all above inequalities into (2.3) and using the definition of χ_k, we deduce that

$$\begin{aligned}\frac{d}{dt}\|\Theta(u)\|_1 + \frac{1}{2}(\boldsymbol{q}, \nabla T_k(u))_\Omega &\le (|f|, |u|\chi_k)_\Omega + (|f|, k(1-\chi_k))_\Omega \\ &\quad \frac{C_2}{2} + C\|\boldsymbol{v} g(u)\|_{q'}^{q'}.\end{aligned}$$

Integration with respect to time then directly leads to (2.1). □

Based on the estimate stated in Lemma 2.1 we can deduce the first a priori estimate

Lemma 2.2. *Let u be a "sufficiently" smooth solution to* (1.1) *or* (1.2) *then the following inequalities hold for all $k \in \mathbb{R}_+$*

$$\begin{aligned}&\sup_{t\in(0,T)} \|u(t)\|_1 + \int_Q |\nabla T_k(u)|^q + |\boldsymbol{q}\chi_k|^{q'}\, dx\, dt \\ &\qquad \le C(k)\left(1 + \int_Q |f| + |\boldsymbol{v}|^{q'}\, dx\, dt + \|u_0\|_1\right)\end{aligned} \tag{2.4}$$

and in the steady case

$$\int_\Omega |\nabla T_k(u)|^q + |\boldsymbol{q}\chi_k|^{q'}\, dx \le C(k)\left(1 + \int_\Omega |f| + |\boldsymbol{v}|^{q'}\, dx\right) \tag{2.5}$$

Proof. The proof is a simple consequence of (2.1), the definition of χ, the continuity of g (that means $\|g(u)\chi_k\|_\infty \le C(k)$), the coercivity (A4) and the fact that for $k \ge 1$

$$|u| - 1 \le \Theta_1(u) \le \Theta_k(u) \le C(k)|u|.$$

□

This Lemma gives first uniform a priori estimates for “bad” data. We see that to control the right hand side, we just need that the initial condition u_0 is bounded in $L^1(\Omega)$, the right hand side f is bounded in $L^1(\mathcal{O})$ and that $\boldsymbol{v}$ is bounded in $L^{q'}(\mathcal{O})$. However, the uniform estimates do not exclude the possibility that u if infinite almost everywhere in steady case. To avoid such situation, we improve the estimates in the following way.

Lemma 2.3. *Let fulfil* (2.4) *or* (2.5). *Then for all* $\lambda > 1$

$$\begin{aligned}\int_Q \frac{|\nabla u|^q + |\boldsymbol{q}|^{q'}}{(1+|u|)^\lambda}\,dx\,dt \\ \le \frac{C}{\lambda-1}\left(1 + \int_Q |f| + \frac{|\boldsymbol{v}|^{q'}|g(u)|^{q'}}{(1+|u|)^\lambda}\,dx\,dt + \|u_0\|_1\right)\end{aligned} \tag{2.6}$$

and in the steady case

$$\int_\Omega \frac{|\nabla u|^q + |\boldsymbol{q}|^{q'}}{(1+|u|)^\lambda}\,dx \le \frac{C}{\lambda-1}\left(1 + \int_\Omega |f| + \frac{|\boldsymbol{v}|^{q'}|g(u)|^{q'}}{(1+|u|)^\lambda}\,dx\right). \tag{2.7}$$

Proof. We again prove (2.6) since the proof of (2.7) is the same. First, we multiply (2.1) by $(1+k)^{-1-\lambda}$ with $\lambda > 0$ and then integrate with respect to k over $(0,\infty)$ to get

$$\begin{aligned}&\int_0^\infty \int_\Omega \frac{\Theta_k(u(t))}{(1+k)^{1+\lambda}}\,dx\,dk + \int_0^\infty \int_0^t \frac{(\boldsymbol{q}, \nabla u \chi_k)_\Omega}{(1+k)^{1+\lambda}}\,d\tau\,dk \\ &\le C\int_0^\infty \int_0^t \int_\Omega \frac{|f||u|\chi_k}{(1+k)^{1+\lambda}} + \frac{|\boldsymbol{v}|^{q'}|g(u)|^{q'}\chi_k}{(1+k)^{1+\lambda}}\,dx\,d\tau\,dk \\ &\quad + C\int_0^\infty \int_0^t \int_\Omega \frac{k|f|(1-\chi_k)}{(1+k)^{1+\lambda}}\,dx\,d\tau\,dk + C\int_0^\infty \int_\Omega \frac{1+\Theta_k(u_0)}{(1+k)^{1+\lambda}}\,dx\,dk.\end{aligned} \tag{2.8}$$

Next, we evaluate all terms. For this purpose, we introduce the following identities. For arbitrary $q : \mathcal{O} \to \mathbb{R}$ there holds

$$\begin{aligned}\int_0^\infty \int_{\mathcal{O}} \frac{g(z)\chi_k(u(z))}{(1+k)^{1+\lambda}}\,dz\,dk &= \int_{\mathcal{O}} \int_0^\infty \frac{g(z)\chi_k(u(z))}{(1+k)^{1+\lambda}}\,dk\,dz \\ &= \int_{\mathcal{O}} \int_{|u(z)|}^\infty \frac{g(z)}{(1+k)^{1+\lambda}}\,dk\,dz = \frac{1}{\lambda}\int_{\mathcal{O}} \frac{g(z)}{(1+|u(z)|)^\lambda}\,dz\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}\int_0^\infty \int_{\mathcal{O}} \frac{kg(z)(1-\chi_k(u(z)))}{(1+k)^{1+\lambda}}\,dz\,dk &= \int_{\mathcal{O}} \int_0^\infty \frac{kg(z)(1-\chi_k(u(z)))}{(1+k)^{1+\lambda}}\,dk\,dz \\ = \int_{\mathcal{O}} \int_0^{|u(z)|} \frac{kg(z)}{(1+k)^{1+\lambda}}\,dk\,dz &= \frac{1}{(1-\lambda)\lambda}\int_{\mathcal{O}} g(z)\frac{1+\lambda|u(z)|}{(1+|u(z)|)^\lambda} - g(z)\,dz\end{aligned} \tag{2.10}$$

Thus, using these identities in (2.8) we get

$$\begin{aligned}\int_0^t \frac{(\boldsymbol{q},\nabla u)_\Omega}{\lambda(1+|u|)^\lambda}\,d\tau &\le C\int_0^t\int_\Omega \frac{|f||u|}{\lambda(1+|u|)^\lambda}+\frac{|\boldsymbol{v}|^{q'}|g(u)|^{q'}}{\lambda(1+|u|)^\lambda}\,dx\,d\tau\\ &\quad + C\int_0^t\int_\Omega \frac{|f|}{\lambda(1-\lambda)}\left(\frac{1+\lambda|u|}{(1+|u|)^\lambda}-1\right)dx\,d\tau\\ &\quad + C\int_0^\infty\int_\Omega \frac{1+\Theta_k(u_0)}{(1+k)^{1+\lambda}}\,dx\,dk-\int_0^\infty\int_\Omega\frac{\Theta_k(u(t))}{(1+k)^{1+\lambda}}\,dx\,dk.\end{aligned}\tag{2.11}$$

Finally, we evaluate the term with Θ.

$$\begin{aligned}\int_0^\infty\int_\Omega\frac{\Theta_k(v)}{(1+k)^{1+\lambda}}\,dx\,dk &= \int_\Omega\int_0^{|v(x)|}\int_0^\infty\frac{T_k(s)}{(1+k)^{1+\lambda}}\,dk\,ds\,dx\\ &=\int_\Omega\int_0^{|v(x)|}\left(\int_0^s\frac{k}{(1+k)^{1+\lambda}}\,dk+\int_s^\infty\frac{s}{(1+k)^{1+\lambda}}\,dk\right)ds\,dx\\ &=\int_\Omega\int_0^{|v(x)|}\frac{1}{\lambda(1-\lambda)}((1+s)^{1-\lambda}-1)\,ds\,dx\\ &=\int_\Omega\frac{(1+|v(x)|)^{2-\lambda}-1}{\lambda}-\frac{|v(x)|}{\lambda(1-\lambda)}\,dx\end{aligned}$$

Hence, inserting this expression into (2.11), assuming that $\lambda>1$ and neglecting non-positive terms on the right hand side, we finally obtain

$$\begin{aligned}(\lambda-1)\int_0^t\frac{(\boldsymbol{q},\nabla u)_\Omega}{(1+|u|)^\lambda}\,d\tau &\le C\int_0^t\int_\Omega\frac{|\boldsymbol{v}|^{q'}|g(u)|^{q'}}{(1+|u|)^\lambda}+|f|\,dx\,d\tau\\ &\quad + C\int_\Omega 1+|u_0|\,dx.\end{aligned}\tag{2.12}$$

Hence, using (A4) for lower bound of the term on the left hand side, we immediately get (2.6). □

2.2. **Further estimates for proper g and $\boldsymbol{v}$.** All estimates derived in the previous subsection still depends on $\boldsymbol{v}$ and $g(u)$, while the dependence on f and u_0 reduces only to the assumption on their integrability. This section is devoted to reduction the dependence of all estimates only on the proper integrability of $\boldsymbol{v}$ and on the growth assumptions for g. First, it evidently follows from Lemma 2.3 that the minimal requirement is $\boldsymbol{v}\in L^{q'}$ and the corresponding estimates are the following.

Lemma 2.4. *Let u fulfil (2.4) or (2.5), g satisfy (1.11) with $p>\frac{1}{q}$. Then there holds*

$$\begin{aligned}&\sup_{t\in(0,T)}\|u(t)\|_1+\int_Q\frac{|\nabla u|^q+|\boldsymbol{q}|^{q'}}{(1+|u|)^{qp}}\,dx\,dt\\ &\qquad\le C(p,q)\left(1+\int_Q|f|+|\boldsymbol{v}|^{q'}\,dx\,dt+\|u_0\|_1\right)\end{aligned}\tag{2.13}$$

and in the steady case

$$\int_\Omega\frac{|\nabla u|^q+|\boldsymbol{q}|^{q'}}{(1+|u|)^{qp}}\,dx\le C(q,p)\left(1+\int_\Omega|f|+|\boldsymbol{v}|^{q'}\,dx\right).\tag{2.14}$$

The restriction $p>\frac{1}{q}$ can be in fact omitted. We can always increase p such that it is valid.

Proof. The proof easily follows from Lemma 2.2 and Lemma 2.3, where we set $\lambda := qp$. $\square$

At this moment, we see that there is the key difference between the steady and the unsteady case. While in the unsteady case, we can claim that u is finite almost everywhere due to the estimate on $\|u(t)\|_1$, the estimate in steady case is much weaker. However, in case that

$$p \le 1, \tag{2.15}$$

that means we assume that the term with g behaves at most like $|u|^{q-1}$, i.e., the critical growth, we can deduce from (2.14) that

$$\int_\Omega |\nabla \ln(1+|u|)|^q \, dx \le C\left(1 + \int_\Omega |f| + |\boldsymbol{v}|^{q'} \, dx\right). \tag{2.16}$$

Note that (2.16) still implies that $|u|$ is finite almost everywhere and therefore gives chance to introduce some (very weak) notion of a solution. On the other hand, we see that in the steady case, we are not able to show finiteness of u in the supercritical case $p > 1$. Note that it could be possible under some structural conditions on g completed by the very weak assumptions on the sign of $\operatorname{div} \boldsymbol{v}$, which we however do not discuss here.

We continue with improving the estimates for u in terms of better assumptions on $\boldsymbol{v}$. For simplicity and for the sake of clarity we focus only on the case $q < d$, since the opposite case can be handled in a similar manner. Also we treat only the case $pq > 1$ since for the opposite case, we already obtained the optimal estimates.

Lemma 2.5 (Improved estimates - steady case)**.** *Let Ω be a Lipschitz domain, u fulfil* (2.5) *and g satisfy* (1.11) *with $p < 1$ and $pq > 1$. Then for all $\lambda \in (1, pq)$ there holds*

$$\int_\Omega |u|^{\frac{d(q-\lambda)}{d-q}} + \frac{|\nabla u|^q + |\boldsymbol{q}|^{q'}}{(1+|u|)^\lambda} \, dx \le C(p, q, \lambda, \|f\|_1, \|\boldsymbol{v}\|_{z(\lambda)}), \tag{2.17}$$

where,

$$z(\lambda) := \frac{d}{q-1} \frac{q-\lambda}{(d-q)(1-p) + q - \lambda}.$$

Note that the constant C in (2.17) explodes as $p \nearrow 1$ or $\lambda \searrow 1$.

Proof. We start the proof with the steady case. Using (2.7) and (1.11) we get

$$\int_\Omega \frac{|\nabla u|^q + |\boldsymbol{q}|^{q'}}{(1+|u|)^\lambda} \, dx \le \frac{C}{\lambda - 1}\left(1 + \int_\Omega |f| + |\boldsymbol{v}|^{q'} + |\boldsymbol{v}|^{q'} |u|^{pq-\lambda} \, dx\right). \tag{2.18}$$

Hence, if $1 < \lambda < q$, we can use the fact that u has zero trace on $\partial\Omega$ and consequently with the help of the the Sobolev embedding we get (here we assume that $q < d$ otherwise we would write down the L^∞ estimate)

$$\begin{aligned}
\|(1+|u|)^{1-\frac{\lambda}{q}}\|^q_{\frac{dq}{d-q}} &\le C(1 + \|(1+|u|)^{1-\frac{\lambda}{q}} - 1\|^q_{\frac{dq}{d-q}}) \\
&\le C(1 + \|\nabla(1+|u|)^{1-\frac{\lambda}{q}}\|^q_q) \\
&= C(\lambda, q)\left(1 + \int_\Omega \frac{|\nabla u|^q}{(1+|u|)^\lambda} \, dx\right).
\end{aligned} \tag{2.19}$$

Thus, using this in (2.18) we gain for $\lambda \in (1, pq]$

(2.20)

$$
\begin{aligned}
\|(1+|u|)^{1-\frac{\lambda}{q}}\|^q_{\frac{dq}{d-q}} &\le C(\lambda)\left(1+\|f\|_1+\int_\Omega |\boldsymbol{v}|^{q'}\left((1+|u|)^{1-\frac{\lambda}{q}}\right)^{\frac{q(pq-\lambda)}{q-\lambda}}\,dx\right)\\
&\le C(\lambda)\left(1+\|f\|_1+\|\boldsymbol{v}\|^{q'}_{\frac{d}{q-1}\frac{q-\lambda}{d-dp+pq-\lambda}}\|(1+|u|)^{1-\frac{\lambda}{q}}\|^{\frac{q(pq-\lambda)}{q-\lambda}}_{\frac{dq}{d-q}}\right),
\end{aligned}
$$

where we used the Hölder inequality. Thus, applying the Young inequality, we get

$$
\|(1+|u|)^{1-\frac{\lambda}{q}}\|^q_{\frac{dq}{d-q}} \le C(\lambda)\left(1+\|f\|_1+\|\boldsymbol{v}\|^{\frac{q-\lambda}{(q-1)(1-p)}}_{\frac{d}{q-1}\frac{q-\lambda}{d-dp+pq-\lambda}}\right). \tag{2.21}
$$

Therefore, combining (2.21) with (2.18), we deduce (2.17). □

Next, since the proof of the following Lemma is in fact identical to the proof of Lemma 2.5 we formulate it without the proof.

Lemma 2.6 (Improved estimates - unsteady case)**.** *Let Ω be a Lipschitz domain, u fulfil (2.4) and g satisfy (1.11) with $p<1$ and $pq>1$. Then for all $\lambda \in (1,pq)$ there holds*

$$
\begin{aligned}
\int_0^T \|(1+|u|)^{1-\frac{\lambda}{q}}\|^q_{\frac{dq}{d-q}}\,dt + \int_Q \frac{|\nabla u|^q+|\boldsymbol{q}|^{q'}}{(1+|u|)^\lambda}\,dx\,dt&\\
\le C(p,q,\lambda,\|f\|_1,Y(\boldsymbol{v},\lambda),\|u_0\|_1),&
\end{aligned} \tag{2.22}
$$

where,

$$
Y(\boldsymbol{v},\lambda) := \int_0^T \|\boldsymbol{v}\|^{\frac{q-\lambda}{(q-1)(1-p)}}_{\frac{d}{q-1}\frac{q-\lambda}{(d-q)(1-p)+q-\lambda}}\,dt.
$$

Finally, we focus on the unsteady case and show that in some cases we are able to treat also the supercritical case $p \ge 1$.

Lemma 2.7 (Improved estimates - unsteady case with $p \ge 1$)**.** *Let Ω be a Lipschitz domain, u fulfil (2.5) and g satisfy (1.11) with $1 \le p < \frac{d+1}{d}$ and $pq>1$. Then for all $\lambda \in (1,pq)$ there holds*

$$
\int_\Omega |u|^{\frac{d(q-\lambda)}{d-q}} + \frac{|\nabla u|^q+|\boldsymbol{q}|^{q'}}{(1+|u|)^\lambda}\,dx \le C(p,q,\lambda,\|f\|_1,\|\boldsymbol{v}\|_{z(\lambda)q'},\|u_0\|_1), \tag{2.23}
$$

where,

$$
z(\lambda) := 1 + \frac{(pq-\lambda)d}{q(d+1-dp)}. \tag{2.24}
$$

Proof. We again start with (2.6) and assuming that $\lambda \in (1,q)$ we can deduce that similarly as above that

$$
\begin{aligned}
\int_0^T \|(1+|u|)^{\frac{q-\lambda}{q}}\|^q_{\frac{dq}{d-q}}\,dt + \int_Q \frac{|\nabla u|^q+|\boldsymbol{q}|^{q'}}{(1+|u|)^\lambda}\,dx\,dt&\\
\le C(\|f\|_1,\|u_0\|_1) + C\int_Q |\boldsymbol{v}|^{q'}(1+|u|)^{pq-\lambda}\,dx\,dt.&
\end{aligned} \tag{2.25}
$$

Our goal is to estimate the last term for which we use the already obtained information

$$
\sup_t \|u(t)\|_1 \le C(\|f\|_1,\|u_0\|_1,\|\boldsymbol{v}\|_{q'}).
$$

Note here that since we assume that $\lambda < q$ the immediately $pq - \lambda > 0$. Consequently, for any $\alpha \in [0,1]$ we may estimate the last term on the right hand side of (2.25) with the help of the Hölder inequality as follows.

$$\begin{aligned}
&\int_Q |\boldsymbol{v}|^{q'}(1+|u|)^{pq-\lambda}\,dx\,dt \\
&= \int_Q |\boldsymbol{v}|^{q'}(1+|u|)^{\alpha(pq-\lambda)}\left((1+|u|)^{\frac{q-\lambda}{q}}\right)^{\frac{q}{q-\lambda}(1-\alpha)(pq-\lambda)}\,dx\,dt \\
&\le C\int_0^T \|\boldsymbol{v}\|_{zq'}^{q'}\|1+|u|\|_1^{\alpha(pq-\lambda)}\|(1+|u|)^{\frac{q-\lambda}{q}}\|_{\frac{dq}{d-q}}^{\frac{q}{q-\lambda}(1-\alpha)(pq-\lambda)}\,dt
\end{aligned} \tag{2.26}$$

provided that

$$\frac{1}{z} + \alpha(pq-\lambda) + \frac{(1-\alpha)(pq-\lambda)}{q-\lambda}\frac{d-q}{d} \le 1. \tag{2.27}$$

Finally, applying the Young inequality, we see that (2.26) reduces to

$$\begin{aligned}
\int_Q |\boldsymbol{v}|^{q'}(1+|u|)^{pq-\lambda}\,dx\,dt &\le C\int_0^T \|\boldsymbol{v}\|_{zq'}^{zq'}\|1+|u|\|_1^{\alpha z(pq-\lambda)}\,dt \\
&\quad + \varepsilon\int_0^T \|(1+|u|)^{\frac{q-\lambda}{q}}\|_{\frac{dq}{d-q}}^{\frac{z'q}{q-\lambda}(1-\alpha)(pq-\lambda)}\,dt.
\end{aligned} \tag{2.28}$$

Hence, for a given $z \in [1,\infty)$, setting α such that

$$\frac{z'}{q-\lambda}(1-\alpha)(pq-\lambda) = 1, \tag{2.29}$$

we get

$$\begin{aligned}
\int_Q |\boldsymbol{v}|^{q'}(1+|u|)^{pq-\lambda}\,dx\,dt &\le C\int_0^T \|\boldsymbol{v}\|_{zq'}^{zq'}\|1+|u|\|_1^{\alpha z(pq-\lambda)}\,dt \\
&\quad + \varepsilon\int_0^T \|(1+|u|)^{\frac{q-\lambda}{q}}\|_{\frac{dq}{d-q}}^{q}\,dt,
\end{aligned} \tag{2.30}$$

and therefore the last term can be absorbed by the first term in (2.25) and we are directly led to (2.23) using the L^1 estimate for u. All we need is to find such z that (2.27) is satisfied. First, note that assuming $z > 1$ and since $p \ge 1$ then $\alpha \in [0,1]$. Substituting (2.29) into (2.27), we see that it is equivalent to find z fulfilling

$$(pq-\lambda)d \le (z-1)q(1+d-pd). \tag{2.31}$$

Since $\lambda < q$ the left hand side is nonnegative and therefore we see the restriction $p < 1 + \frac{1}{d}$ otherwise the right hand side would be negative. Then an elementary algebraic manipulation leads to (2.24). $\square$

We end this section by proving the final estimates of all quantities appearing in the problem (1.1) or in (1.2). For simplicity, we consider here, the most optimistic case, i.e., the case when the integrability of $\boldsymbol{v}$ is good enough and when we can control the natural quantity appearing in all estimates for all $\lambda > 1$. If it would not be the case, we could simply use the same procedure, but with weaker results. First, we consider the steady case.

Lemma 2.8. *Let $\boldsymbol{q}$ and u (being zero on $\partial\Omega$) fulfill for all $\lambda > 1$*

$$\int_\Omega \frac{|\nabla u|^q + |\boldsymbol{q}|^{q'}}{(1+|u|)^\lambda}\,dx \le C(\lambda). \tag{2.32}$$

Then for all $0 < \varepsilon \ll 1$ the following holds

$$\int_\Omega (1+|u|)^{\frac{d(q-1)}{d-q}-\varepsilon} + |\nabla u|^{\frac{d(q-1)}{d-1}-\varepsilon} + |\boldsymbol{q}|^{\frac{d}{d-1}-\varepsilon} \le C(\varepsilon^{-1},\Omega). \tag{2.33}$$

Proof. Similarly as above, we can deduce from (2.32) and from the fact that u has zero trace that

$$\|(1+|u|)^{\frac{q-\lambda}{q}}\|_{\frac{dq}{d-q}} \le C(\lambda),$$

which directly leads to the first estimate in (2.33), since λ can be taken arbitrarily closed to one. Second, taking $p < q$ and $\tilde{p} < q'$ arbitrary we can deduce by the Young inequality that

$$\int_\Omega |\nabla u|^p = \int_\Omega \left(\frac{|\nabla u|^q}{(1+|u|)^\lambda}\right)^{\frac{p}{q}} (1+|u|)^{\frac{p\lambda}{q}} \le C(\lambda) + \int_\Omega (1+|u|)^{\frac{p\lambda}{q-p}}$$

$$\int_\Omega |\boldsymbol{q}|^{\tilde{p}} = \int_\Omega \left(\frac{|\boldsymbol{q}|^{q'}}{(1+|u|)^\lambda}\right)^{\frac{\tilde{p}}{q'}} (1+|u|)^{\frac{\tilde{p}\lambda}{q'}} \le C(\lambda) + \int_\Omega (1+|u|)^{\frac{\tilde{p}\lambda}{q'-\tilde{p}}},$$

where we also used the assumption (2.32). Hence, to control the last term on the right hand sides, we use the first part of the estimate (2.33) and we see that we need to chose p, $\tilde{p}$ and λ such that

$$\frac{p\lambda}{q-p} < \frac{d(q-1)}{d-q} \qquad \Leftrightarrow \qquad p < \frac{d(q-1)}{d-1},$$
$$\frac{\tilde{p}\lambda}{q'-\tilde{p}} < \frac{d(q-1)}{d-q} \qquad \Leftrightarrow \qquad \tilde{p} < \frac{d}{d-1}.$$

The equivalence follows from the fact that λ can be chosen arbitrarily close to one. Consequently, combining such a choice of parameters we gain the rest of the estimate (2.33). $\square$

Second, we focus on the unsteady case.

Lemma 2.9. *Let $\boldsymbol{q}$ and u (being zero on $\partial\Omega$) fulfill for all $\lambda > 1$*

$$\sup_{t\in(0,T)} \|u(t)\|_1 + \int_Q \frac{|\nabla u|^q + |\boldsymbol{q}|^{q'}}{(1+|u|)^\lambda}\,dx\,dt \le C(\lambda). \tag{2.34}$$

Then for all $0 < \varepsilon \ll 1$ the following hold

(2.35)
$$\int_Q (1+|u|)^{q-\frac{d-q}{d}-\varepsilon} + |\nabla u|^{q-\frac{d}{d+1}-\varepsilon} + |\boldsymbol{q}|^{1+\frac{q'-1}{d+1}-\varepsilon} \le C(\varepsilon^{-1}) \quad \textit{if } q > \frac{2d}{d+1},$$
$$\int_Q |\nabla u|^{\frac{q}{2}-\varepsilon} + |\boldsymbol{q}|^{\frac{q'}{2}-\varepsilon} \le C(\varepsilon^{-1}) \quad \textit{if } q \le \frac{2d}{d+1}.$$

Proof. Similarly as above, we can deduce from (2.34) and from the fact that u has zero trace that

$$\int_0^T \|(1+|u|)^{\frac{q-\lambda}{q}}\|^q_{\frac{dq}{d-q}}\,dt \le C(\lambda). \tag{2.36}$$

Hence, we see that if $q \le \frac{2d}{d+1}$ we did not improve any integrability with respect to the spatial variable and the best estimate we have on u is that one coming from (2.34). Therefore in this case, it is rather straightforward to observe (2.35). Indeed, with the help of the Young inequality we get

$$\int_Q |\nabla u|^{\frac{q}{\lambda+1}} = \int_Q \left(\frac{|\nabla u|^q}{(1+|u|)^\lambda}\right)^{\frac{1}{\lambda+1}} (1+|u|)^{\frac{\lambda}{\lambda+1}} \le C(\lambda) + \int_Q (1+|u|),$$

and also

$$\int_Q |\boldsymbol{q}|^{\frac{q'}{\lambda+1}} = \int_Q \left(\frac{|\boldsymbol{q}|^{q'}}{(1+|u|)^\lambda}\right)^{\frac{1}{\lambda+1}} (1+|u|)^{\frac{\lambda}{\lambda+1}} \le C(\lambda) + \int_Q (1+|u|).$$

Thus using (2.34) we directly get the second part of (2.35). On the other hand, if $q > \frac{2d}{d+1}$ we can interpolate (2.36) and (2.34) to improve the integrability of u with respect to the spatial variable as follows

$$\begin{aligned}
\int_0^T \|u\|_z^z &\le \int_0^T \|(1+|u|)^{\frac{q-\lambda}{q}}\|_{\frac{zq}{q-\lambda}}^{\frac{zq}{q-\lambda}} \\
&\le \int_0^T \|(1+|u|)^{\frac{q-\lambda}{q}}\|_{\frac{q}{q-\lambda}}^{(1-\alpha)\frac{zq}{q-\lambda}} \|(1+|u|)^{\frac{q-\lambda}{q}}\|_{\frac{dq}{d-q}}^{\frac{\alpha zq}{q-\lambda}} \\
&\overset{(2.34)}{\le} C\int_0^T \|(1+|u|)^{\frac{q-\lambda}{q}}\|_{\frac{dq}{d-q}}^{\frac{\alpha zq}{q-\lambda}} \overset{(2.36)}{\le} C(\lambda),
\end{aligned}$$

provided that $\frac{\alpha z}{q-\lambda} \le 1$. Here, the interpolation is given by the formula (for $z \in (1, \frac{d(q-\lambda)}{d-q})$)

$$\frac{q-\lambda}{zq} = \frac{(1-\alpha)(q-\lambda)}{q} + \frac{\alpha(d-q)}{dq}. \tag{2.37}$$

Thus, if we choose the optimal α, i.e., $\alpha := \frac{q-\lambda}{z}$, and substitute it into (2.37) we get

$$z = q - \lambda + \frac{q}{d}, \tag{2.38}$$

which directly lead to the first part of the estimate (2.35) if we let $\lambda = 1+\varepsilon$. Next, we focus on estimates on ∇u and also $\boldsymbol{q}$. We proceed as above. Hence, taking $p < q$ and $\tilde{p} < q'$ arbitrary we can deduce by the Young inequality that

$$\int_\Omega |\nabla u|^p = \int_\Omega \left(\frac{|\nabla u|^q}{(1+|u|)^\lambda}\right)^{\frac{p}{q}} (1+|u|)^{\frac{p\lambda}{q}} \le C(\lambda) + \int_\Omega (1+|u|)^{\frac{p\lambda}{q-p}}$$

$$\int_\Omega |\boldsymbol{q}|^{\tilde{p}} = \int_\Omega \left(\frac{|\boldsymbol{q}|^{q'}}{(1+|u|)^\lambda}\right)^{\frac{\tilde{p}}{q'}} (1+|u|)^{\frac{\tilde{p}\lambda}{q'}} \le C(\lambda) + \int_\Omega (1+|u|)^{\frac{\tilde{p}\lambda}{q'-\tilde{p}}},$$

where we also used the assumption (2.34). Hence, to control the last term on the right hand sides, we use the first part of the estimate (2.35) and we see that we need to chose p, $\tilde{p}$ and λ such that

$$\begin{aligned}
\frac{p\lambda}{q-p} &< q - 1 + \frac{q}{d} \qquad \Leftrightarrow \qquad p < q - \frac{d}{d+1}, \\
\frac{\tilde{p}\lambda}{q'-\tilde{p}} &< q - 1 + \frac{q}{d} \qquad \Leftrightarrow \qquad \tilde{p} < 1 + \frac{q'-1}{d+1}.
\end{aligned}$$

The equivalence follows from the fact that λ can be chosen arbitrarily close to one. Consequently, combining such a choice of parameters we gain the rest of the estimate (2.35). $\square$

3. How to solve the problem

3.1. **Various notions of solution.** In the previous section we provided a lot of various estimates depending on the regularity of $\boldsymbol{v}$. This gave us a hint in which spaces we should look for a solution and we also observed that u may not necessarily be a Sobolev (Bochner-Sobolev) function. Moreover, we also realized that not all quantities in the studied equation are for such estimates integrable. Therefore, we employ here two different notion of solution, both of them consistent with the concept of the classical solution provided that the it itself is smoother. Hence, the first set of definitions is related to a general notion of a weak solution.

Definition 3.1 (weak-steady). *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz set. Assume that that the maximal monotone graph $\mathcal{A}$ satisfies (A1)–(A5). Assume that $f \in \mathcal{M}(\Omega)$ and that $\boldsymbol{g}$ is given by* (1.9) *with some measurable $\boldsymbol{v}$ and $\boldsymbol{v}_{\mathrm{div}}$. We say that a couple $(u, \boldsymbol{q})$ is a weak solution to* (1.2) *if $u : \Omega \to \mathbb{R}$ and $\boldsymbol{q} : \Omega \to \mathbb{R}^d$ are measurable functions such that*

$$u \text{ is finite almost everywhere in } \Omega, \tag{3.1}$$

$$\boldsymbol{q} \in L^1(\Omega; \mathbb{R}^d), \tag{3.2}$$

$$T_k(u) \in W_0^{1,q}(\Omega) \qquad \text{for all } k \in \mathbb{R}_+, \tag{3.3}$$

$$(\nabla T_k(u), T_k'(u)\boldsymbol{q}) \in \mathcal{A}(\cdot, u) \qquad \text{for all } k \in \mathbb{R}_+ \text{ and a.e. in } \Omega, \tag{3.4}$$

$$\boldsymbol{g}(x, u(x)) \in L^1(\Omega; \mathbb{R}^d), \tag{3.5}$$

and satisfy

$$(\boldsymbol{q}, \nabla\varphi)_\Omega = -(\boldsymbol{g}(u), \nabla\varphi)_\Omega + \langle f, \varphi\rangle_\Omega \qquad \text{for all } \varphi \in W_0^{1,\infty}(\Omega). \tag{3.6}$$

Definition 3.2 (weak-unsteady). *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz set. Assume that that the maximal monotone graph $\mathcal{A}$ satisfies (A1)–(A5). Assume that $f \in \mathcal{M}(Q)$, $e_0 \in \mathcal{M}$ and that $\boldsymbol{g}$ is given by* (1.9) *with some measurable $\boldsymbol{v}$ and $\boldsymbol{v}_{\mathrm{div}}$. We say that a couple $(u, \boldsymbol{q})$ is a weak solution to* (1.1) *if $u : Q \to \mathbb{R}$ and $\boldsymbol{q} : \Omega \to \mathbb{R}^d$ are measurable functions such that*

$$u \in L^1(Q), \tag{3.7}$$

$$\boldsymbol{q} \in L^1(Q; \mathbb{R}^d), \tag{3.8}$$

$$T_k(u) \in L^q(0, T; W_0^{1,q}(\Omega)) \qquad \text{for all } k \in \mathbb{R}_+, \tag{3.9}$$

$$(\nabla T_k, T_k'(u)\boldsymbol{q}) \in \mathcal{A}(\cdot, u) \qquad \text{for all } k \in \mathbb{R}_+ \text{ and a.e. in } \in Q, \tag{3.10}$$

$$\boldsymbol{g}(t, x, u(t, x)) \in L^1(Q; \mathbb{R}^d), \tag{3.11}$$

and satisfy

$$\begin{aligned} &-(u, \varphi_{,t})_Q + (\boldsymbol{q}, \nabla\varphi)_Q = -(\boldsymbol{g}(u), \nabla\varphi)_Q + \langle f, \varphi\rangle_Q + \langle e_0, \varphi(0)\rangle_\Omega \\ &\qquad\qquad \text{for all } \varphi \in \mathcal{D}(-\infty, T; \mathcal{C}^\infty(\overline{\Omega})). \end{aligned} \tag{3.12}$$

As we saw there are lot of restrictions to get the existence of a weak solution. A possibility how to avoid such restriction is introduce a different concept of solution - the entropy solution. However, the prise we pay for such a definition is that we are not able to include the data in measures but we must impose the assumptions on their integrability.

Definition 3.3 (entropy-steady)**.** *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz set. Assume that that the maximal monotone graph $\mathcal{A}$ satisfies (A1)–(A5). Assume that $f \in L^1(\Omega)$ and that $\boldsymbol{g}$ is given by* (1.9) *with some measurable $\boldsymbol{v} \in L^{q'}(\Omega)$ and $\boldsymbol{v}_{\mathrm{div}} \in L^1(\Omega)$. We say that a couple $(u, \boldsymbol{q})$ is an entropy solution to* (1.2) *if $u : \Omega \to \mathbb{R}$ and $\boldsymbol{q} : \Omega \to \mathbb{R}^d$ are measurable functions such that*

$$u \text{ is finite almost everywhere in } \Omega, \tag{3.13}$$

$$\boldsymbol{q}T_k'(u) \in L^q(\Omega;\mathbb{R}^d) \qquad \text{for all } k \in \mathbb{R}_+, \tag{3.14}$$

$$T_k(u) \in W_0^{1,q}(\Omega) \qquad \text{for all } k \in \mathbb{R}_+, \tag{3.15}$$

$$(\nabla T_k(u), T_k'(u)\boldsymbol{q}) \in \mathcal{A}(\cdot, u) \qquad \text{for all } k \in \mathbb{R}_+ \text{ and a.e. in } \Omega, \tag{3.16}$$

and satisfy

$$\begin{aligned} &(\boldsymbol{q}, \nabla T_k(u-\varphi))_\Omega \le (\boldsymbol{v}g(u), \nabla T_k(u-\varphi))_\Omega + (f, T_k(u-\varphi))_\Omega \\ &\quad - (\boldsymbol{v}_{\mathrm{div}}, \nabla\varphi T_k(u-\varphi))_\Omega \qquad \text{for all } \varphi \in W_0^{1,\infty}(\Omega). \end{aligned} \tag{3.17}$$

Definition 3.4 (entropy-unsteady)**.** *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz set. Assume that that the maximal monotone graph $\mathcal{A}$ satisfies (A1)–(A5). Assume that $f \in L^1(Q)$, $e_0 \in L^1(\Omega)$ and that $\boldsymbol{g}$ is given by* (1.9) *with some measurable $\boldsymbol{v} \in L^{q'}(Q)$ and $\boldsymbol{v}_{\mathrm{div}} \in L^1(Q)$. We say that a couple $(u, \boldsymbol{q})$ is an entropy solution to* (1.1) *if $u : Q \to \mathbb{R}$ and $\boldsymbol{q} : \Omega \to \mathbb{R}^d$ are measurable functions such that*

$$u \in L^\infty(0,T;L^1(\Omega)), \tag{3.18}$$

$$\boldsymbol{q}T_k'(u) \in L^1(Q;\mathbb{R}^d) \qquad \text{for all } k \in \mathbb{R}_+, \tag{3.19}$$

$$T_k(u) \in L^q(0,T;W_0^{1,q}(\Omega)) \qquad \text{for all } k \in \mathbb{R}_+, \tag{3.20}$$

$$(\nabla T_k, T_k'(u)\boldsymbol{q}) \in \mathcal{A}(\cdot, u) \qquad \text{for all } k \in \mathbb{R}_+ \text{ and a.e. in } \in Q, \tag{3.21}$$

and satisfy

(3.22)

$$\begin{aligned} &\int_\Omega \Theta_k(u(t)-\varphi(t))\,dx + \int_0^t (\varphi_t, T_k(u-\varphi))_\Omega + (\boldsymbol{q}, \nabla T_k(u-\varphi))_\Omega + (\boldsymbol{v}_{\mathrm{div}}, \nabla\varphi T_k(u-\varphi))\,d\tau \\ &\le \int_0^t (\boldsymbol{v}g(u), \nabla T_k(u-\varphi))_\Omega + (f, T_k(u-\varphi))_\Omega\,d\tau + \int_\Omega \Theta_k(u_0-\varphi(0)) \\ &\qquad \text{for all } \varphi \in \mathcal{D}(-\infty,T;\mathcal{C}^\infty(\overline{\Omega})). \end{aligned}$$

Consistency of a definition

3.2. **Results - existence & uniqueness.** [5] First, we focus on the existence of a weak solution. This however requires that we must impose good enough information on the data so that all integrals are well defined. We do not provide here the complete list of compatible assumptions but rather we keep the assumptions in such a way that one can still use the estimates that do not differ from the standard ones in the theory of elliptic equations with measure right hand side.

[5] All statements studied in this notes hold for f of the form $f = f_1 + f_2$, where $f_1 \in \mathcal{M}(Q)$ (or $f \in L^1(Q)$) and $f_2 \in L^{q'}(0,T;W^{-1,q'}(\Omega))$. This easy generalization is omitted, for simplicity.

Theorem 3.1 (Weak solution - steady)**.** *Let all assumptions of Definition 3.1 be satisfied. Moreover, assume that*

$$q > \frac{2d}{d+1}, \tag{3.23}$$

$$p < 1, \tag{3.24}$$

$$\boldsymbol{v}_{\mathrm{div}} \in L^r(\Omega;\mathbb{R}^d) \qquad \textit{for } r > \frac{d(q-1)}{d(q-1)-d+q}, \tag{3.25}$$

$$\boldsymbol{v} \in L^s(\Omega;\mathbb{R}^d) \qquad \textit{for } s > \frac{d}{d(1-p)+pq-1}. \tag{3.26}$$

Then there exists a weak solution to (1.2)*.*

Theorem 3.2 (Weak solution - unsteady)**.** *Let all assumptions of Definition 3.2 be satisfied. Moreover, let*

$$q > \frac{2d}{d+1}, \tag{3.27}$$

$$p < 1 + \frac{1}{d}, \tag{3.28}$$

$$\boldsymbol{v}_{\mathrm{div}} \in L^r(Q;\mathbb{R}^d) \qquad \textit{for } r > 1 + \frac{d}{dq-2d+q}, \tag{3.29}$$

$$\boldsymbol{v} \in L^s(Q;\mathbb{R}^d) \qquad \textit{for } s > 1 + \frac{d(pq-1)}{q(d+1-dp)}. \tag{3.30}$$

Then there exists a weak solution to (1.1)*.*

Theorem 3.3 (Entropy solution - steady)**.** *Let all assumptions of Definition 3.3 be satisfied. Moreover, assume that*

$$p \le 1. \tag{3.31}$$

Then there exists an entropy solution to (1.2)*. Moreover, if the graph $\mathcal{A}$ is independent of u and strictly monotone, $\boldsymbol{v} \equiv 0$ and $\boldsymbol{v}_{\mathrm{div}} \in L^{q'}(\Omega)$ then the entropy solution is unique.*

Theorem 3.4 (Entropy solution - unsteady)**.** *Let all assumptions of Definition 3.4 be satisfied. Then there exists an entropy solution to* (1.1)*. Moreover, if the graph $\mathcal{A}$ is independent of u, $\boldsymbol{v} \equiv 0$ and $\boldsymbol{v}_{\mathrm{div}} \in L^{q'}(\Omega)$ then the entropy solution is unique.*

4. Proofs

We show the existence of a solution by a proper approximative scheme. In addition, the use of the same approximation is also a part of the uniqueness proof for the entropy solution. Thus, for given f, u_0, $\boldsymbol{v}$ and $\boldsymbol{v}_{\mathrm{div}}$ we introduce their smooth approximations $\{f^n, u_0^n, \boldsymbol{v}^n, \boldsymbol{v}_{\mathrm{div}}^n\}_{n=1}^{\infty}$ fulfilling

$$\begin{aligned}
\boldsymbol{v}^n &\to \boldsymbol{v} && \text{strongly in } L^r(\mathcal{O};\mathbb{R}^d),\\
\boldsymbol{v}_{\mathrm{div}}^n &\to \boldsymbol{v}_{\mathrm{div}} && \text{strongly in } L^s(\mathcal{O};\mathbb{R}^d), \quad \operatorname{div} \boldsymbol{v}_{\mathrm{div}}^n = 0,\\
f^n &\rightharpoonup^* f && \text{weakly}^* \text{ in } \mathcal{M}(\mathcal{O}),\\
u_0^n &\rightharpoonup^* u_0 && \text{weakly}^* \text{ in } \mathcal{M}(\Omega),
\end{aligned} \tag{4.1}$$

or in case that $f \in L^1(\mathcal{O})$ and $u_0 \in L^1(\Omega)$

$$
\begin{aligned}
f^n &\to f \quad \text{strongly in } L^1(\mathcal{O}),\\
u_0^n &\to u \quad \text{strongly in } L^1(\mathcal{O}).
\end{aligned} \tag{4.2}
$$

For such regular data, we define an approximate problem to (1.1) as: for each $n \in \mathbb{N}$ find $(u^n, \boldsymbol{q}^n)$ such that

$$
\begin{aligned}
u^n_{,t} + \operatorname{div}\left(\boldsymbol{v}^n_{\mathrm{div}} u^n + \frac{n\boldsymbol{v}^n g(u^n)}{n + |g(u^n)|}\right) - \operatorname{div} \boldsymbol{q} = f^n \qquad & \text{in } Q,\\
(\nabla u^n, \boldsymbol{q}^n) \in \mathcal{A}(\cdot, u^n) \qquad & \text{in } Q,\\
u = 0 \qquad & \text{on } (0,T) \times \partial\Omega,\\
u^n(0) = u_0^n \qquad & \text{in } \Omega.
\end{aligned} \tag{4.3}
$$

Similarly for the stationary problem (1.2) we state a problem: for each $n \in \mathbb{N}$ find $(u^n, \boldsymbol{q}^n)$ such that

$$
\begin{aligned}
\operatorname{div}\left(\boldsymbol{v}^n_{\mathrm{div}} u^n + \frac{n\boldsymbol{v}^n g(u^n)}{n + |g(u^n)|}\right) - \operatorname{div} \boldsymbol{q} = f^n \qquad & \text{in } \Omega,\\
(\nabla u^n, \boldsymbol{q}^n) \in \mathcal{A}(\cdot, u^n) \qquad & \text{in } \Omega,\\
u = 0 \qquad & \text{on } \partial\Omega.
\end{aligned} \tag{4.4}
$$

The existence of a weak solution to (4.3) and (4.4) in case when $\boldsymbol{q}$ is a continuous function of ∇u can be proven by using the monotone operator theory and compact embedding theorems (Sobolev embedding or Aubin-Lions lemma) and goes back to Minty. In case we work with the maximal monotone graphs, one could in fact also quote Minty who was surely aware of such possible application. On the other hand the first relevant study for the maximal monotone graph setting is goes to ..... For more recent results on the theory for maximal monotone graph setting with a general (non-polynomial) growth assumptions we refer for example to ....

4.1. **Existence - steady case.** For both existence theorems, i.e., Theorem 3.1 and Theorem 3.3, we use the approximation (4.4) and consider that for each n there is a couple $(u^n, \boldsymbol{q}^n)$ such that $u \in W^{1,q}_0(\Omega)$ and $\boldsymbol{q} \in L^{q'}(\Omega; \mathbb{R}^d)$ fulfilling

$$
\begin{aligned}
-\left(\boldsymbol{v}^n_{\mathrm{div}} u^n + \frac{n\boldsymbol{v}^n g(u^n)}{n + |g(u^n)|}, \nabla\psi\right)_\Omega + (\boldsymbol{q}, \nabla\psi)_\Omega = (f^n, \psi)_\Omega \quad & \text{for all } \psi \in W^{1,p}_0(\Omega),\\
(\nabla u^n, \boldsymbol{q}^n) \in \mathcal{A}(\cdot, u^n) \quad & \text{in } \Omega.
\end{aligned} \tag{4.5}
$$

At this level of approximation, we may use u^n as a test function and therefore we may employ the estimates from Section 2. Consequently, using Lemma 2.2, we get from (2.5) and from the fact that $r \ge q'$ in (4.1) that

$$
\|T_k(u^n)\|_{1,p} + \|\boldsymbol{q}^n T_k'(u^n)\|_{q'} \le C(k), \tag{4.6}
$$

where $C(k)$ is independent of n. Moreover, since in both cases (weak solution or entropy solution) $p < 1$, we deduce from Lemma 2.4 that

$$
\int_\Omega \frac{|\nabla u^n|^q + |\boldsymbol{q}^n|^{q'}}{(1 + |u^n|)^q} \le C. \tag{4.7}
$$

Moreover, a direct consequence of (4.7) (see (2.16)) is

$$
\|(\operatorname{sign} u^n) \ln(1 + |u^n|)\|_{1,q} \le C. \tag{4.8}
$$

In addition, in case we deal with the weak solution, we also use Lemma 2.5 and the assumption (3.26). Indeed, we want to use (2.17), where $z(\lambda)$-norm appear. However, we see that for all relevant $\lambda > 1$ we have

$$z(\lambda) \le \frac{d}{(d-q)(1-p)+q-1} \le r,$$

where r is given by (3.26) and (4.1). Thus, since Ω is bounded, we also have $\|\boldsymbol{v}^n\|_{z(\lambda)} \le C\|\boldsymbol{v}^n\|_r \le C$ and it follows from the assumption (3.26) and Lemma 2.5 that in case we are interested in weak solutions we also have

$$\int_\Omega \frac{|\nabla u^n|^q + |\boldsymbol{q}^n|^{q'}}{(1+|u|^n)^\lambda} \le C(\lambda) \qquad \text{for all } \lambda > 1 \tag{4.9}$$

and also from Lemma 2.8 we get

$$\int_\Omega |u^n|^{\frac{d(q-1)}{d-q}-\varepsilon} + |\boldsymbol{q}^n|^{\frac{d}{d-1}-\varepsilon} \le C(\varepsilon) \qquad \text{for all } \varepsilon > 0. \tag{4.10}$$

First convergence results. Having such a priori estimates in reflexive spaces we can extract a subsequence that we do not relabel and using also the compact embedding we find that

$$\omega^n := (\operatorname{sign} u^n)\ln(1+|u^n|) \rightharpoonup \omega \qquad \text{weakly in } W_0^{1,q}(\Omega), \tag{4.11}$$

$$\omega^n \to \omega \qquad \text{strongly in } L^q(\Omega), \tag{4.12}$$

$$\omega^n \to \omega \qquad \text{almost everywhere in } \Omega. \tag{4.13}$$

Next, due to the strict monotonicity of $(\operatorname{sign} s)\ln(1+|s|)$ we can find a uniquely defined measurable u such that

$$(\operatorname{sign} u)\ln(1+|u|) := \omega \tag{4.14}$$

and using also (4.13) we deduce that

$$u^n \to u \qquad \text{almost everywhere in } \Omega \tag{4.15}$$

and from (4.6) we also have

$$T_k(u^n) \rightharpoonup T_k(u) \qquad \text{weakly in } W_0^{1,q}(\Omega). \tag{4.16}$$

In order to identify also the limit object $\boldsymbol{q}$, we define

$$\boldsymbol{\eta}^n := \frac{\boldsymbol{q}^n}{(1+|u^n|)^{q-1}}$$

and using (4.7) we gain

$$\boldsymbol{\eta}^n \rightharpoonup \boldsymbol{\eta} \qquad \text{weakly in } L^{q'}(\Omega;\mathbb{R}^d) \tag{4.17}$$

and we define the limit object $\boldsymbol{q}$ as

$$\boldsymbol{q} := (1+|u|)^{q-1}\boldsymbol{\eta}. \tag{4.18}$$

Then it directly follows from (4.17)–(4.18) that for all $b \in \mathcal{C}_0(\mathbb{R})$ we have

$$\boldsymbol{q}^n b(u^n) \rightharpoonup \boldsymbol{q} b(u) \qquad \text{weakly in } L^{q'}(\Omega;\mathbb{R}^d). \tag{4.19}$$

Moreover, in case we work with weak solution, we can use (4.10) and it follows from (4.15) and (4.19) that

$$\boldsymbol{q}^n \rightharpoonup \boldsymbol{q} \qquad \text{weakly in } L^{\frac{d}{d-1}-\varepsilon}(\Omega;\mathbb{R}^d), \tag{4.20}$$

$$u^n \to u \qquad \text{strongly in } L^{\frac{d(q-1)}{d-q}-\varepsilon}(\Omega) \tag{4.21}$$

for all $\varepsilon > 0$.

Finally, we focus on the limit $n \to \infty$ in (4.5). First, in case we want to deal with the weak solution we need to show that

$$\boldsymbol{v}^n_{\mathrm{div}} u^n \to \boldsymbol{v}_{\mathrm{div}} u \qquad \text{strongly in } L^1(\Omega;\mathbb{R}^d), \tag{4.22}$$

$$\frac{n\boldsymbol{v}^n g(u^n)}{n+|g(u^n)|} \to \boldsymbol{v} g(u) \qquad \text{strongly in } L^1(\Omega). \tag{4.23}$$

To do so, we first observe as a direct consequence of (4.1) and (4.15) that the limits in (4.22)–(4.23) are attained point-wisely almost everywhere in Ω. Thus to show (4.22)–(4.23), it is enough to prove that for some $\delta > 0$ we have

$$\int_\Omega |\boldsymbol{v}^n_{\mathrm{div}} u^n|^{1+\delta} + |\boldsymbol{v} g(u^n)|^{1+\delta} \le C \tag{4.24}$$

and then by using the Vitali theorem we can complete the proof of (4.22)–(4.23). The uniform estimate (4.24) is however the direct consequence of the assumptions (3.25) and (3.26), the convergence assumption (4.1) and the Hölder inequality.

Limit passage. Now, we have everything prepared for letting $n \to \infty$ in $(4.5)_1$ to obtain (3.6). Indeed, for $\psi \in W^{1,\infty}_0(\Omega)$, we can identify the limit in the first term with the help of (4.22) and (4.23). The limit procedure in the second term, follows from the weak convergence result (4.20) and the limit on the right hand side of (4.5) can be easily identified by using the assumption $(4.1)_3$. This in fact ends the proof of Theorem 3.1 provided we show (3.4). This we postpone to the end of this subsection and we proceed further with the entropy solution. For this purpose, we set $\psi := T_k(u^n - \varphi)$ in (4.5) with arbitrary $\varphi \in W^{1,\infty}_0(\Omega)$. Note that at this level of approximation it is a legal test function due to the regularity of u^n. Hence, we obtain

$$\begin{aligned} -\left(\boldsymbol{v}^n_{\mathrm{div}} u^n + \frac{n\boldsymbol{v}^n g(u^n)}{n+|g(u^n)|}, \nabla T_k(u^n-\varphi)\right)_\Omega &+ (\boldsymbol{q}, \nabla T_k(u^n-\varphi))_\Omega \\ = (f^n, T_k(u^n-\varphi))_\Omega. \end{aligned} \tag{4.25}$$

To identify the limits, we first rewrite the first and the third term as follows

$$\begin{aligned} -(\boldsymbol{v}^n_{\mathrm{div}} u^n, \nabla T_k(u^n-\varphi))_\Omega &= -(\boldsymbol{v}^n_{\mathrm{div}} (u^n-\varphi), \nabla T_k(u^n-\varphi))_\Omega \\ &\quad - (\boldsymbol{v}^n_{\mathrm{div}} \varphi, \nabla T_k(u^n-\varphi))_\Omega \\ &= -(\boldsymbol{v}^n_{\mathrm{div}}, \nabla \Theta_k(u^n-\varphi))_\Omega - (\boldsymbol{v}^n_{\mathrm{div}} \varphi, \nabla T_k(u^n-\varphi))_\Omega \\ &= (\operatorname{div} \boldsymbol{v}^n_{\mathrm{div}}, \Theta_k(u^n-\varphi))_\Omega + (\operatorname{div}(\boldsymbol{v}^n_{\mathrm{div}} \varphi), T_k(u^n-\varphi))_\Omega \\ &= (\boldsymbol{v}^n_{\mathrm{div}}, T_k(u^n-\varphi)\nabla\varphi)_\Omega, \end{aligned}$$

where we used integration by parts and the fact that $\operatorname{div} \boldsymbol{v}^n_{\mathrm{div}} = 0$. For the third term we use the following

$$(\boldsymbol{q}^n, \nabla T_k(u^n-\varphi))_\Omega = (\boldsymbol{q}^n - \boldsymbol{q}^*(u^n,\nabla\varphi), \nabla T_k(u^n-\varphi))_\Omega + (\boldsymbol{q}^*(u^n,\nabla\varphi), \nabla T_k(u^n-\varphi))_\Omega,$$

where $\boldsymbol{q}^*$ denotes the selection from the assumption (A5). Note that from the monotonicity of the graph $\mathcal{A}(u^n)$ it follows that the first term is nonnegative. Using these to identities, we can rewrite (4.25) in the way

$$\begin{aligned}(\boldsymbol{v}^n_{\mathrm{div}}, T_k(u^n-\varphi)\nabla\varphi)_\Omega &- \left(\frac{n\boldsymbol{v}^n g(u^n)}{n+|g(u^n)|}, \nabla T_k(u^n-\varphi)\right)_\Omega \\ &+ (\boldsymbol{q}^n - \boldsymbol{q}^*(u^n,\nabla\varphi), \nabla T_k(u^n-\varphi))_\Omega + (\boldsymbol{q}^*(u^n,\nabla\varphi), \nabla T_k(u^n-\varphi))_\Omega \\ &= (f^n, T_k(u^n-\varphi))_\Omega.\end{aligned} \tag{4.26}$$

It remains to identify all limits in (4.26). First, we notice that the point-wise convergence (4.15) leads to

$$T_k(u^n-\varphi) \rightharpoonup^* T_k(u-\varphi) \qquad \text{weakly}^* \text{ in } L^\infty(\Omega). \tag{4.27}$$

Therefore, using also (4.2), it is not difficult to show that (note that here we need the strong convergence of f^n in L^1 and the convergence in the space of measures would not be sufficient)

$$(f^n, T_k(u^n-\varphi))_\Omega \to (f, T_k(u-\varphi))_\Omega.$$

Similarly, since φ is Lipschitz, it follows from (4.27) and (4.1) that (here the main advantage of the entropy formulation appears, since wee se we need the compactnes of $\boldsymbol{v}^n_{\mathrm{div}}$ only in the space L^1)

$$(\boldsymbol{v}^n_{\mathrm{div}}, T_k(u^n-\varphi)\nabla\varphi)_\Omega \to (\boldsymbol{v}_{\mathrm{div}}, T_k(u-\varphi)\nabla\varphi)_\Omega.$$

In addition, since φ is Lipschitz it follows that there exists $M>0$ such that

$$|\nabla T_k(u^n-\varphi)| \le C T'_M(u^n)(1+|\nabla T_M(u^n)|).$$

Consequently, using (4.15) and (4.16), we see that

$$T_k(u^n-\varphi) \rightharpoonup T_k(u-\varphi) \qquad \text{weakly in } W^{1,q}_0(\Omega), \tag{4.28}$$

$$\frac{ng(u^n)\nabla T_k(u^n-\varphi)}{n+|g(u^n)|} \rightharpoonup g(u)\nabla T_k(u-\varphi) \qquad \text{weakly in } L^q(\Omega;\mathbb{R}^d). \tag{4.29}$$

Thus using also the continuity of the selection with respect to u, the fact that φ is Lipschitz and the assumption (4.1) (which means that $\boldsymbol{v}^n$ is compact in $L^{q'}$), we see that

$$\begin{aligned}\left(\frac{n\boldsymbol{v}^n g(u^n)}{n+|g(u^n)|}, \nabla T_k(u^n-\varphi)\right)_\Omega &\to (\boldsymbol{v}g(u), \nabla T_k(u-\varphi))_\Omega \\ (\boldsymbol{q}^*(u^n,\nabla\varphi), \nabla T_k(u^n-\varphi))_\Omega &\to (\boldsymbol{q}^*(u,\nabla\varphi), \nabla T_k(u-\varphi))_\Omega\end{aligned}$$

Consequently, assuming that

$$\limsup_{n\to\infty} (\boldsymbol{q}^n - \boldsymbol{q}^*(u^n,\nabla\varphi), \nabla T_k(u^n-\varphi))_\Omega \ge (\boldsymbol{q} - \boldsymbol{q}^*(u,\nabla\varphi), \nabla T_k(u-\varphi))_\Omega, \tag{4.30}$$

we can let $n\to\infty$ in (4.26) to deduce (3.17). Hence, it remains to check (4.30) and also (3.16).

Identification of the graph. The rest of this section is devoted to the identification of the most critical nonlinearity $\boldsymbol{q}$, namely to show the validity of (3.4) or (3.16). Then the proof of (4.30) for entropy solution will follow. For this purpose, we use the Lipschitz approximation method and a technique based on the renormalization of (4.4). We start with the renormalization property of (4.4). First, recall the definition of the truncation function T_k in (1.12) and its mollification $T_{k,\delta}$, see (1.13). To do so, we set $\psi := T'_{k,\delta}(u^n)\varphi$ in (4.5) with arbitrary $\varphi \in W_0^{1,q}(\Omega)$. Doing so, we observe that for all $\varphi \in W_0^{1,q}(\Omega)$ the following identity holds

$$
\begin{aligned}
&-\left(\boldsymbol{v}^n_{\mathrm{div}} u^n T'_{k,\delta}(u^n) + \frac{nT'_{k,\delta}(u^n)\boldsymbol{v}^n g(u^n)}{n+|g(u^n)|}, \nabla\varphi\right)_\Omega \\
&\quad -\left(\boldsymbol{v}^n_{\mathrm{div}} u^n + \frac{n\boldsymbol{v}^n g(u^n)}{n+|g(u^n)|}, \varphi\nabla T'_{k,\delta}(u^n)\right)_\Omega \\
&\quad +(\boldsymbol{q}^n T'_{k,\delta}(u^n), \nabla\varphi)_\Omega + (\boldsymbol{q}^n, \varphi\nabla T'_{k,\delta}(u^n))_\Omega = (f^n T'_{k,\delta}(u^n), \varphi)_\Omega
\end{aligned}
\tag{4.31}
$$

Next, we rearrange the second term by using the fact that $\operatorname{div} \boldsymbol{v}_{\mathrm{div}} = 0$ na integration by parts as

$$
\begin{aligned}
-\left(\boldsymbol{v}^n_{\mathrm{div}} u^n, \varphi\nabla T'_{k,\delta}(u^n)\right)_\Omega &= \left(\boldsymbol{v}^n_{\mathrm{div}} u^n, T'_{k,\delta}(u^n)\nabla\varphi\right)_\Omega + \left(\boldsymbol{v}^n_{\mathrm{div}}\varphi, T'_{k,\delta}(u^n)\nabla u^n\right)_\Omega \\
&= \left(\boldsymbol{v}^n_{\mathrm{div}} u^n, T'_{k,\delta}(u^n)\nabla\varphi\right)_\Omega + (\boldsymbol{v}^n_{\mathrm{div}}\varphi, \nabla T_{k,\delta}(u^n))_\Omega \\
&= \left(\boldsymbol{v}^n_{\mathrm{div}} u^n, T'_{k,\delta}(u^n)\nabla\varphi\right)_\Omega - (\boldsymbol{v}^n_{\mathrm{div}} T_{k,\delta}(u^n), \nabla\varphi)_\Omega .
\end{aligned}
$$

Hence, substituting this into (4.31) we deduce

$$
\begin{aligned}
(\boldsymbol{q}^n T'_{k,\delta}(u^n), \nabla\varphi)_\Omega &= \left(\frac{nT'_{k,\delta}(u^n)\boldsymbol{v}^n g(u^n)}{n+|g(u^n)|}, \nabla\varphi\right)_\Omega + (\boldsymbol{v}^n_{\mathrm{div}} T_{k,\delta}(u^n), \nabla\varphi)_\Omega \\
&\quad - (\boldsymbol{q}^n\varphi, \nabla T'_{k,\delta}(u^n))_\Omega + (f^n T'_{k,\delta}(u^n), \varphi)_\Omega \\
&\quad + \left(\frac{n\boldsymbol{v}^n g(u^n)}{n+|g(u^n)|}, \varphi\nabla T'_{k,\delta}(u^n)\right)_\Omega .
\end{aligned}
\tag{4.32}
$$

We focus on the limiting procedure in (4.32). Using the assumption (4.1), the pointwise convergence (4.15), the boundedness of $T_{k,\delta}$ and the fact that $T'_{k,\delta} \in \mathcal{C}_0(\mathbb{R})$ we can deduce

$$
\frac{nT'_{k,\delta}(u^n)\boldsymbol{v}^n g(u^n)}{n+|g(u^n)|} \to T'_{k,\delta}(u)\boldsymbol{v} g(u) \qquad \text{strongly in } L^{q'}(\Omega;\mathbb{R}^d), \tag{4.33}
$$

$$
\boldsymbol{v}^n_{\mathrm{div}} T_{k,\delta}(u^n) \to \boldsymbol{v}_{\mathrm{div}} T_{k,\delta}(u) \qquad \text{strongly in } L^1(\Omega;\mathbb{R}^d). \tag{4.34}
$$

Moreover, using (4.1) and (4.6), we can for all k, δ extract a subsequence that we do not relabel such that

$$
\boldsymbol{q}^n \cdot \nabla T'_{k,\delta}(u^n) \rightharpoonup^* \mu_1 \qquad \text{weakly}^* \text{ in } \mathcal{M}(\Omega), \tag{4.35}
$$

$$
f^n T'_{k,\delta}(u^n) \rightharpoonup^* \mu_2 \qquad \text{weakly}^* \text{ in } \mathcal{M}(\Omega), \tag{4.36}
$$

$$
\frac{n\boldsymbol{v}^n g(u^n)}{n+|g(u^n)|} \cdot \nabla T'_{k,\delta}(u^n) \rightharpoonup^* \mu_3 \qquad \text{weakly}^* \text{ in } \mathcal{M}(\Omega). \tag{4.37}
$$

Note that here we do not claim that we are able to identify the measures μ_i but we just want to use these results to let $n \to \infty$ in (4.32). Indeed, using (4.33)–(4.37) and also (4.19), it follows from (4.32) that for all $\varphi \in W_0^{1,\infty}(\Omega)$ (note that

$W_0^{1,\infty}(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega})$)

$$(\boldsymbol{q}T'_{k,\delta}(u), \nabla\varphi)_\Omega = \left(T'_{k,\delta}(u)\boldsymbol{v}g(u), \nabla\varphi\right)_\Omega + (\boldsymbol{v}_{\mathrm{div}}T_{k,\delta}(u), \nabla\varphi)_\Omega - \langle\mu_1, \varphi\rangle_\Omega + \langle\mu_2, \varphi\rangle_\Omega + \langle\mu_3, \varphi\rangle_\Omega. \tag{4.38}$$

Consequently, let $\{w^n\}_{n=1}^\infty$ be an arbitrary sequence fulfilling

$$w^n \rightharpoonup^* w \qquad \text{weakly}^* \text{ in } W^{1,\infty}(\Omega), \tag{4.39}$$

$$w^n \to w \qquad \text{strongly in } \mathcal{C}(\overline{\Omega}). \tag{4.40}$$

Then due to the (4.19) we can extract a subsequence such that

$$\boldsymbol{q}^n T'_{k,\delta}(u^n) \cdot \nabla w^n \rightharpoonup \mu_4 \qquad \text{weakly in } L^{q'}(\Omega). \tag{4.41}$$

Our first particular goal is to show that

$$\mu_4 = \boldsymbol{q}T'_{k,\delta}(u) \cdot \nabla w \qquad \text{almost everywhere in } \Omega. \tag{4.42}$$

For this purpose we define $z^n := w^n\eta$ with arbitrary $\eta \in \mathcal{C}_0^1(\Omega)$ and easily observe from (4.39) and (4.40) that for $z := w\eta$

$$z^n \rightharpoonup^* z \qquad \text{weakly}^* \text{ in } W_0^{1,\infty}(\Omega), \tag{4.43}$$

$$z^n \to z \qquad \text{strongly in } \mathcal{C}_0(\overline{\Omega}). \tag{4.44}$$

Thus setting $\varphi := z^n$ in (4.32) we find that

$$\begin{aligned}\lim_{n\to\infty} (\boldsymbol{q}^n T'_{k,\delta}(u^n), \nabla z^n)_\Omega &= \lim_{n\to\infty} \left(\frac{nT'_{k,\delta}(u^n)\boldsymbol{v}^n g(u^n)}{n + |g(u^n)|}, \nabla z^n\right)_\Omega \\ &\quad + \lim_{n\to\infty} (\boldsymbol{v}^n_{\mathrm{div}} T_{k,\delta}(u^n), \nabla z^n)_\Omega \\ &\quad - \lim_{n\to\infty} (\boldsymbol{q}^n z^n, \nabla T'_{k,\delta}(u^n))_\Omega + (f^n T'_{k,\delta}(u^n), z^n)_\Omega \\ &\quad + \lim_{n\to\infty} \left(\frac{n\boldsymbol{v}^n g(u^n)}{n + |g(u^n)|}, z^n \nabla T'_{k,\delta}(u^n)\right)_\Omega .\end{aligned} \tag{4.45}$$

However, combining the convergence results (4.33)–(4.37) and (4.43)–(4.44), we see that it is not difficult to identify limits on the right hand side of (4.45) to conclude that

$$\lim_{n\to\infty} (\boldsymbol{q}^n T'_{k,\delta}(u^n), \nabla z^n)_\Omega = \left(T'_{k,\delta}(u)\boldsymbol{v}g(u), \nabla z\right)_\Omega + (\boldsymbol{v}_{\mathrm{div}} T_{k,\delta}(u), \nabla z)_\Omega - \langle\mu_1, z\rangle_\Omega + \langle\mu_2, z\rangle_\Omega + \langle\mu_3, z\rangle_\Omega.$$

Thus setting also $\varphi := z$ in (4.38) and comparing the both results we obtain

$$\lim_{n\to\infty} (\boldsymbol{q}^n T'_{k,\delta}(u^n), \nabla z^n)_\Omega = (\boldsymbol{q}T'_{k,\delta}(u), \nabla z)_\Omega, \tag{4.46}$$

which by using the definition of z^n and z leads to

$$\begin{aligned}\lim_{n\to\infty} (\boldsymbol{q}^n T'_{k,\delta}(u^n), \nabla w^n \eta)_\Omega &= (\boldsymbol{q}T'_{k,\delta}(u), \nabla w\eta)_\Omega \\ &\quad + \lim_{n\to\infty} (\boldsymbol{q}T'_{k,\delta}(u)w - \boldsymbol{q}^n T'_{k,\delta}(u^n)w^n, \nabla\eta)_\Omega \\ &= (\boldsymbol{q}T'_{k,\delta}(u), \nabla w\eta)_\Omega,\end{aligned} \tag{4.47}$$

where for the second equality we used (4.19) and (4.40). Thus, since μ_4 unique, the equality (4.42) directly follows from (4.47).

Next, we strengthen (4.41)–(4.42) such that they hold also for w^n fulfilling only

$$w^n \rightharpoonup w \qquad \text{weakly in } W^{1,q}(\Omega). \tag{4.48}$$

Of course such a result cannot be valid but for this reason we restrict ourselves onto the smaller parts of Ω. For this purpose, we use the Biting lemma, see Lemma A.1. Indeed, using (4.19), we see that

$$\int_\Omega |\boldsymbol{q}^n T'_{k,\delta}(u^n)|^{q'} \le C(k,\delta)$$

and consequently we can find a nondecreasing sequence of measurable sets $\Omega_\ell \subset \Omega_{\ell+1} \subset \Omega$ fulfilling

$$|\Omega \setminus \Omega_\ell| \overset{\ell\to\infty}{0},$$

such that for each ℓ there exists a subsequence that we do not relabel

$$|\boldsymbol{q}^n T'_{k,\delta}(u^n)|^{q'} \rightharpoonup \mu_5 \qquad \text{weakly in } L^1(\Omega_\ell) \tag{4.49}$$

and consequently due to the equivalent characterization of a weakly compact sets in L^1 we see that for all $\varepsilon > 0$ there exists $h > 0$ such that for all $\tilde{\Omega} \subset \Omega_\ell$ fulfilling $|\tilde{\Omega}| < h$ there holds

$$\int_{\tilde{\Omega}} |\boldsymbol{q}^n T'_{k,\delta}(u^n)|^{q'} \le \varepsilon. \tag{4.50}$$

Our next goal is to show that for arbitrary sequence w^n fulfilling (4.48) we can for each $\ell \in \mathbb{N}$ extract a subsequence such that

$$\boldsymbol{q}^n T'_{k,\delta}(u^n) \cdot \nabla w^n \rightharpoonup \boldsymbol{q} T'_{k,\delta}(u) \cdot \nabla w \qquad \text{weakly in } L^1(\Omega_\ell). \tag{4.51}$$

To do so, we apply the Lipschtiz approximation method, see Lemma B.1 and find w^n_λ and $\Omega^n_\lambda := \{x \in \Omega;\ M|\nabla w^n| > \lambda\}$ such that

$$\|w^n_\lambda\|_{1,\infty} \le C(\lambda), \tag{4.52}$$

$$w^n_\lambda = \boldsymbol{w}^n \text{ in } \Omega \setminus \Omega^n_\lambda, \tag{4.53}$$

$$\|w^n_\lambda\|_{1,q} \le C\|w^n\|_{1,q}. \tag{4.54}$$

Therefore we can extract a subsequence and find w_λ (note that w_λ is not given by the Lipschitz approximation lemma, but it is just a weak limit of the sequence) such that

$$w^n_\lambda \rightharpoonup^* w_\lambda \qquad \text{in } W^{1,\infty}(\Omega).$$

Therefore, we may apply (4.42) to conclude

$$\boldsymbol{q}^n T'_{k,\delta}(u^n) \cdot \nabla w^n_\lambda \rightharpoonup \boldsymbol{q} T'_{k,\delta}(u) \cdot \nabla w_\lambda \qquad \text{weakly in } L^{q'}(\Omega)$$

and consequently also

$$\boldsymbol{q}^n T'_{k,\delta}(u^n) \cdot \nabla w^n_\lambda \rightharpoonup \boldsymbol{q} T'_{k,\delta}(u) \cdot \nabla w_\lambda \qquad \text{weakly in } L^{q'}(\Omega_\ell) \tag{4.55}$$

for each $\ell \in \mathbb{N}$. Finally, we let $\lambda \to \infty$ in (4.55). To do so, we first observe that due to the uniform integrability (4.50) we have

$$\begin{aligned}
\int_{\Omega_\ell} |\boldsymbol{q}^n T'_{k,\delta}(u^n)||\nabla w^n - \nabla w^n_\lambda| &\overset{(4.53)}{=} \int_{\Omega_\lambda \cap \Omega_\ell} |\boldsymbol{q}^n T'_{k,\delta}(u^n)||\nabla w^n - \nabla w^n_\lambda| \\
&\le \left(\int_{\Omega_\lambda \cap \Omega_\ell} |\boldsymbol{q}^n T'_{k,\delta}(u^n)|^{q'} \right)^{\frac{1}{q'}} \|\nabla w^n - \nabla w^n_\lambda\|_q \\
&\le \left(\int_{\Omega_\lambda \cap \Omega_\ell} |\boldsymbol{q}^n T'_{k,\delta}(u^n)|^{q'} \right)^{\frac{1}{q'}}
\end{aligned}$$

Thus, using the definition of Ω_λ and the weak type estimate

$$|\Omega_\lambda| \le \frac{C\|\nabla w^n\|_1}{\lambda} \le \frac{C}{\lambda},$$

we can conclude from the property (4.50) that

$$\limsup_{\lambda\to\infty} \sup_n \int_{\Omega_\ell} |\boldsymbol{q}^n T'_{k,\delta}(u^n)||\nabla w^n - \nabla w^n_\lambda| = 0. \tag{4.56}$$

Moreover, using (4.54) we see that $\|w_\lambda\|_{1,p} \le C$ and therefore we can extract a subsequence (with respect to λ) such that

$$w_\lambda \rightharpoonup \tilde{w} \qquad \text{weakly in } W^{1,q}(\Omega) \tag{4.57}$$

for some $\tilde{w} \in W^{1,q}(\Omega)$. In fact we show that $\tilde{w} = w$. Indeed, using the compact embedding we have that

$$w^n_\lambda \to w_\lambda \qquad \text{strongly in } L^q(\Omega).$$

Consequently,

$$\|w_\lambda - w\|_1 = \lim_{n\to\infty} \|w^n_\lambda - w^n\|_1 = \lim_{n\to\infty} \int_{\Omega^n_\lambda} |w^n_\lambda - w^n| \le C|\Omega_\lambda|^{\frac{1}{q'}} \le \frac{C}{\lambda^{q'}}$$

and therefore

$$\lim_{\lambda\to\infty} \|w_\lambda - w\|_1 = 0.$$

Since the weak limit $\tilde{w}$ is unique, we see that necessarily $\tilde{w} = w$. Finally, combining (4.55)–(4.57), we conclude (4.51).

It remains the last step, namely we choose $w^n := T_k(u^n)$ and consequently $w := T_k(u)$. Note that due to (4.16) it is a possible choice. Hence according to (4.51) there holds

$$\boldsymbol{q}^n T'_{k,\delta}(u^n) \cdot \nabla T_k(u^n) \rightharpoonup \boldsymbol{q} T'_{k,\delta}(u) \cdot \nabla T_k(u) \qquad \text{weakly in } L^1(\Omega_\ell).$$

However, since almost everywhere in Ω we have

$$\boldsymbol{q}^n T'_{k,\delta}(u^n) \cdot \nabla T_k(u^n) = \boldsymbol{q}^n T'_k(u^n) \cdot \nabla T_k(u^n),$$
$$\boldsymbol{q} T'_{k,\delta}(u) \cdot \nabla T_k(u) = \boldsymbol{q} T'_k(u) \cdot \nabla T_k(u)$$

we simply get that

$$\boldsymbol{q}^n T'_k(u^n) \cdot \nabla T_k(u^n) \rightharpoonup \boldsymbol{q} T'_k(u) \cdot \nabla T_k(u) \qquad \text{weakly in } L^1(\Omega_\ell). \tag{4.58}$$

The rest of the proof is based on the maximality of the graph. Let $\boldsymbol{w} \in \mathbb{R}^d$ be arbitrary. Denoting

$$\boldsymbol{u}^n := \boldsymbol{q}^*(\cdot, u^n, \boldsymbol{w}),$$

we have that $\|\boldsymbol{u}^n\|_\infty \le C$ and due to the continuity of $\boldsymbol{q}^*$ with respect to u^n due to the point-wise convergence of u^n (4.15) we deduce that

$$\boldsymbol{u}^n \to \boldsymbol{u} \qquad \text{strongly in } L^{q'}(\Omega; \mathbb{R}^d), \tag{4.59}$$

where $\boldsymbol{u} = \boldsymbol{q}^*(\cdot, u, \boldsymbol{w})$. Thus, taking into account (4.58) and (4.59) we observe

$$(\boldsymbol{q}^n T'_k(u^n) - \boldsymbol{u}^n) \cdot (\nabla T_k(u^n) - \boldsymbol{w}) \rightharpoonup (\boldsymbol{q} T'_k(u) - \boldsymbol{u}) \cdot (\nabla T_k(u) - \boldsymbol{w}) \quad \text{weakly in } L^1(\Omega_\ell). \tag{4.60}$$

But since $(\boldsymbol{q}^n T'_k(u^n), \nabla T_k(u^n))$ and also $(\boldsymbol{u}^n, \boldsymbol{w})$ belong to $\mathcal{A}(u^n)$ almost everywhere, we have that

$$(\boldsymbol{q}^n T'_k(u^n) - \boldsymbol{u}^n) \cdot (\nabla T_k(u^n) - \boldsymbol{w}) \ge 0 \text{ almost everywhere in } \Omega_\ell$$

and it directly follows from (4.60) that also

$$(\boldsymbol{q}T_k'(u) - \boldsymbol{u}) \cdot (\nabla T_k(u) - \boldsymbol{w}) \geq 0 \qquad \text{almost everywhere in } \Omega_\ell). \tag{4.61}$$

Thus, using the maximality of the graph, namely (A6), and the fact that $\boldsymbol{w}$ is arbitrary, we deduce from (4.61) that

$$(\boldsymbol{q}T_k'(u), \nabla T_k(u)) \in \mathcal{A}(u) \qquad \text{almost everywhere in } \Omega_\ell.$$

Finally, since $|\Omega \setminus \Omega_\ell| \to 0$ as $\ell \to \infty$, we gen generalize the above formula to get (3.4) and (3.16). This finishes the proof of the existence of a weak solution and it remains to finish the proof also for the entropy solution, hence we need to show (4.30).

First, it follows from (4.15) and the fact that $\varphi \in W^{1,\infty}(\Omega)$ that

$$T_k(u^n - \varphi) \to T_k(u - \varphi) \qquad \text{strongly in } L^1(\Omega)$$

and due to the Egorof theorem for any $\varepsilon > 0$ there exists Ω_ε fulfilling $|\Omega \setminus \Omega_\varepsilon| \leq \varepsilon$ such that

$$T_k(u^n - \varphi) \to T_k(u - \varphi) \qquad \text{strongly in } \mathcal{C}(\Omega_\varepsilon). \tag{4.62}$$

Next, denoting $\Omega_\delta := \{x \in \Omega;\ |u - \varphi| \leq k - \delta\}$ then it follows from (4.62) that there exists n_0 such that for all $n \geq n_0$ there holds

$$T_k(u^n - \varphi) = u^n - \varphi \text{ in } \Omega_\varepsilon \cap \Omega_\delta. \tag{4.63}$$

Hence, denoting $M := \|\varphi\|_\infty$, using the monotonicity of the graph, we may deduce the lower estimate (we recall the definition of Ω_ℓ in (4.49))

$$\begin{aligned}
&\int_\Omega (\boldsymbol{q}^n - \boldsymbol{q}^*(u^n, \nabla\varphi)) \cdot \nabla T_k(u^n - \varphi) \\
&\quad \geq \int_{\Omega_\varepsilon \cap \Omega_\delta \cap \Omega_\ell} (\boldsymbol{q}^n - \boldsymbol{q}^*(u^n, \nabla\varphi)) \cdot \nabla T_k(u^n - \varphi) \\
&\quad = \int_{\Omega_\varepsilon \cap \Omega_\delta \cap \Omega_\ell} (\boldsymbol{q}^n - \boldsymbol{q}^*(u^n, \nabla\varphi)) \cdot (\nabla u^n - \nabla\varphi) \\
&\quad = \int_{\Omega_\varepsilon \cap \Omega_\delta \cap \Omega_\ell} (\boldsymbol{q}^n T_{M+k}'(u^n) - \boldsymbol{q}^*(u^n, \nabla\varphi)) \cdot (\nabla T_{M+k}(u^n) - \nabla\varphi),
\end{aligned}$$

where the last equality follows from the definition of M and Ω_δ. Thus, using (4.58) and the same procedure as in (4.60) we may deduce that

$$\begin{aligned}
&\limsup_{n\to\infty} \int_\Omega (\boldsymbol{q}^n - \boldsymbol{q}^*(u^n, \nabla\varphi)) \cdot \nabla T_k(u^n - \varphi) \\
&\quad \geq \lim_{n\to\infty} \int_{\Omega_\varepsilon \cap \Omega_\delta \cap \Omega_\ell} (\boldsymbol{q}^n T_{M+k}'(u^n) - \boldsymbol{q}^*(u^n, \nabla\varphi)) \cdot (\nabla T_{M+k}(u^n) - \nabla\varphi) \\
&\quad = \int_{\Omega_\varepsilon \cap \Omega_\delta \cap \Omega_\ell} (\boldsymbol{q}T_{M+k}'(u) - \boldsymbol{q}^*(u, \nabla\varphi)) \cdot (\nabla T_{M+k}(u) - \nabla\varphi) \\
&\quad = \int_{\Omega_\varepsilon \cap \Omega_\ell} (\boldsymbol{q} - \boldsymbol{q}^*(u, \nabla\varphi)) \cdot \nabla T_{k-\delta}(u - \varphi).
\end{aligned} \tag{4.64}$$

Since the integrand is an integrable function, we can let $\delta \to 0_+$, $\varepsilon \to 0_+$ and $\ell \to \infty$ in (4.64) to gain (4.30). The proof is complete.

4.2. **Uniqueness - steady case.** Let us consider that $(\boldsymbol{q}^1, u^1)$ is an arbitrary entropy weak solution and in what follows we show that it coincides with the solution constructed in the previous subsection. First, according to the assumptions $\boldsymbol{v} \equiv 0$ and therefore

$$(\boldsymbol{q}^1, \nabla T_k(u^1 - \varphi))_\Omega \le (f, T_k(u^1-\varphi))_\Omega - (\boldsymbol{v}_{\mathrm{div}}, \nabla\varphi T_k(u^1-\varphi))_\Omega \tag{4.65}$$

for all $\varphi \in W^{1,\infty}_0(\Omega)$. Moreover, since $\boldsymbol{v}_{\mathrm{div}}$ is assumed to belong to $L^{q'}(\Omega;\mathbb{R}^d)$ we can use the density argument and see that (4.65) holds for all $\varphi \in W^{1,q}_0(\Omega)\cap L^\infty(\Omega)$. Next, consider the couple $(\boldsymbol{q}, u)$ obtained by the limit procedure introduced in the previous section, i.e., the limit obtained from the solution of

$$(\boldsymbol{q}^n, \nabla\psi)_\Omega - (\boldsymbol{v}^n_{\mathrm{div}} u^n, \nabla\psi)_\Omega = (f^n, \psi)_\Omega \qquad \text{for all } \psi \in W^{1,q}_0(\Omega). \tag{4.66}$$

Moreover, this solution satisfies the renormalized equation (4.31), which reduces in the case $\boldsymbol{v} = 0$ to

$$\begin{aligned}(\boldsymbol{v}^n_{\mathrm{div}}, \varphi\nabla T_{m,\delta}(u^n))_\Omega &+ (\boldsymbol{q}^n T'_{m,\delta}(u^n), \nabla\varphi)_\Omega + (\boldsymbol{q}^n, \varphi\nabla T'_{m,\delta}(u^n))_\Omega\\ &= (f^n T'_{m,\delta}(u^n), \varphi)_\Omega \qquad \text{for all } \varphi \in L^\infty \cap W^{1,q}_0(\Omega).\end{aligned} \tag{4.67}$$

Finally, we set $\varphi := T_{m,\delta}(u^n)$ in (4.65) and $\varphi := -T_k(u^1 - T_{m,\delta}(u^n))$ in (4.67) and sum the result to obtain

$$\begin{aligned}&(\boldsymbol{q}^1 - \boldsymbol{q}^n T'_{m,\delta}(u^n), \nabla T_k(u^1 - T_{m,\delta}(u^n)))_\Omega\\ &\le (f - f^n T'_{m,\delta}(u^n), T_k(u^1 - T_{m,\delta}(u^n)))_\Omega\\ &\quad - (\boldsymbol{v}_{\mathrm{div}} - \boldsymbol{v}^n_{\mathrm{div}}, \nabla T_{m,\delta}(u^n) T_k(u^1 - T_{m,\delta}(u^n)))_\Omega\\ &\quad + (\boldsymbol{q}^n, T_k(u^1 - T_{m,\delta}(u^n))\nabla T'_{m,\delta}(u^n))_\Omega.\end{aligned} \tag{4.68}$$

Our first, goal is to let $\delta \to 0_+$. Such a procedure is easy in the first three terms but we must check the last one where $T''_{k,\delta}$ appears and may become singular. First, using the Hölder inequality, we have an estimate

$$\begin{aligned}(\boldsymbol{q}^n, T_k(u^1 - T_{m,\delta}(u^n))\nabla T'_{m,\delta}(u^n))_\Omega &\le k(|T''_{m,\delta}(u^n)|\boldsymbol{q}^n, \nabla u^n)_\Omega\\ &= -k(T''_{m,\delta}(|u^n|)\boldsymbol{q}^n, \nabla u^n)_\Omega.\end{aligned} \tag{4.69}$$

Note that the right hand side is nonnegative since $(\boldsymbol{q}^n, \nabla u^n) \in \mathcal{A}(u^n)$. To evaluate the right hand side, we set $\psi := 1 - T'_{m,\delta}(u^n_+)$ in (4.66), where $u^n_+ := \max(0, u^n)$. Note that $\psi \in W^{1,q}_0(\Omega)$. Doing so, we obtain the identity

$$\begin{aligned}-(\boldsymbol{q}^n, T''_{m,\delta}(u^n_+)\nabla u^n)_\Omega &= (f^n, 1 - T'_{m,\delta}(u^n_+))_\Omega - (\boldsymbol{v}^n_{\mathrm{div}} u^n T''_{m,\delta}(u^n_+), \nabla u^n)_\Omega\\ &= (f^n, 1 - T'_{m,\delta}(u^n_+))_\Omega + (\boldsymbol{v}^n_{\mathrm{div}}(T'_{m,\delta}(u^n_+) - 1), \nabla u^n_+)_\Omega\\ &= (f^n, 1 - T'_{m,\delta}(u^n_+))_\Omega + (\boldsymbol{v}^n_{\mathrm{div}}, \nabla(T_{m,\delta}(u^n_+) - u^n_+))_\Omega\\ &= (f^n, 1 - T'_{m,\delta}(u^n_+))_\Omega \le (|f^n|, 1 - T'_{m-1,1}(u^n_+))_\Omega,\end{aligned} \tag{4.70}$$

where we used the integration by parts formula and the inequality $|1 - T'_{m,\delta}(u^n_+)| \le 1 - T'_{m-1,1}(u^n_+)$. By the same procedure, we can obtain the same inequality also for u^n_- and the resulting inequality is of the form

$$-(\boldsymbol{q}^n, T''_{m,\delta}(|u^n|)\nabla u^n)_\Omega \le (|f^n|, 1 - T'_{m-1,1}(u^n))_\Omega. \tag{4.71}$$

In addition, using the Hölder and the triangle inequality, one is directly led to

$$(f - f^n T'_{m,\delta}(u^n), T_k(u^1 - T_{m,\delta}(u^n)))_\Omega \le k\|f^n - f\|_1 + k(|f^n|, 1 - 1 - T'_{m-1,1}(u^n))_\Omega.$$

Hence, substituting these estimates into (4.68) and letting $\delta \to 0_+$ we deduce that

$$\begin{aligned}&(\boldsymbol{q}^1 - \boldsymbol{q}^n T_m'(u^n), \nabla T_k(u^1 - T_m(u^n)))_\Omega \\ &\le k\|f^n - f\| + k(|f^n|, 1 - T_{m-1,1}'(u^n))_\Omega \\ &\quad - (\boldsymbol{v}_{\mathrm{div}} - \boldsymbol{v}^n_{\mathrm{div}}, \nabla T_m(u^n) T_k(u^1 - T_m(u^n)))_\Omega.\end{aligned}$$

In the next step, we let $n \to \infty$. For the left hand side, we use the same procedure as above to pass to the limit with possibly inequality sign. For the first two terms on the right hand side we use the compactness of f^n in L^1 and also the point-wise convergence of u^n and for the last term we use the compactness of $\boldsymbol{v}^n_{\mathrm{div}}$ in $L^{q'}$ and the boundedness of $T_m(u^n)$ in $W^{1,q}$ to gain the resulting inequality

$$(\boldsymbol{q}^1 - \boldsymbol{q} T_m'(u), \nabla T_k(u^1 - T_m(u)))_\Omega \le k(|f|, 1 - T_{m-1,1}'(u))_\Omega. \tag{4.72}$$

Note that due to the fact that $(\boldsymbol{q}^1, u^1)$ and $(\boldsymbol{q}, u)$ are solution we have point-wisely

$$\boldsymbol{q}^1 - \boldsymbol{q} T_m'(u), \nabla T_k(u^1 - T_m(u)) \ge 0.$$

Consequently, letting $m \to \infty$ in (4.72) and recalling that u is finite almost everywhere in Ω we gain

$$\|(\boldsymbol{q}^1 - \boldsymbol{q}) \cdot \nabla T_k(u^1 - u)\|_1 = 0$$

and by passing $k \to \infty$ we see that

$$(\boldsymbol{q}^1 - \boldsymbol{q}) \cdot (\nabla u^1 - \nabla u) \equiv 0 \qquad \text{in } \Omega.$$

However, the graph is assumed to be strictly monotone and since $u^1 = u$ on $\partial\Omega$ we see that $u^1 \equiv u$ almost everywhere and the proof is complete.

4.3. **Existence - unsteady case.**

4.4. **Uniqueness - unsteady case.**

Appendix A. Tools

Lemma A.1. *Let $\{f^n\}_{n=1}^\infty$ be a bounded sequence in $L^1(\mathcal{O})$ with $\mathcal{O}$ bounded measurable set. Then there exists a nondecreasing sequence $\mathcal{O}_\ell \subset \mathcal{O}_{\ell+1} \subset \mathcal{O}$ such that $|\mathcal{O} \setminus \mathcal{O}_\ell| \to 0$ as $\ell \to \infty$ and for each $\ell \in \mathbb{N}$ there exists a subsequence such that*

$$f^n \rightharpoonup f \text{ weakly in } L^1(\mathcal{O}_\ell).$$

Appendix B. Lipschitz approximation of Sobolev functions

Lemma B.1. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. Then there exists a constant C such that for all $\lambda \in \mathbb{R}_+$ and all $w \in W_0^{1,1}(\Omega)$ there exists $w_\lambda \in W_0^{1,\infty}(\Omega)$ fulfilling*

$$|\nabla w_\lambda| \le C\lambda, \tag{B.1}$$

$$w_\lambda = w \text{ in } \Omega \setminus \Omega_\lambda, \tag{B.2}$$

$$\Omega_\lambda := \{x \in \Omega;\ M|\nabla w|(x) > \lambda\}, \tag{B.3}$$

$$\|w_\lambda\|_{1,q} \le C(q)\|w\|_{1,q} \qquad \text{for all } q \in (1, \infty]. \tag{B.4}$$

Proof. The proof is a combination of the results Acerbi and Fusco [1988], Frehse et al. [2003], Diening et al. [2008] □

Appendix C. Lipschitz approximation of Bochner functions

This appendix summarizes the properties of Lipschitz approximations of Bochner functions. We prove only those assertions that extend the results established in Diening et al. [2010] and are available in Bulíček et al. [2012].

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