HOLOMORPHIC DEFORMATIONS AND VECTORIAL CALCULUS OF VARIATIONS.

A STUDY OF THE BURKHOLDER FUNCTIONAL

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Abstract. These lectures are based on joint work with Tadeusz Iwaniec, Istvan Prause and Eero Saksman, and on the papers [7]-[9] in particular.

The aim of the lectures is to discuss topics in the Vectorial calculus of variations, using the methods of holomorphic deformations, but we will also find strong interactions with norm estimates for Singular Integrals, the Beurling transform in particular, and with optimal distortion results in the theory of Quasiconformal Mappings.

1. Introduction and background

A basic question in Calculus of Variations is to characterise the continuous functions $\mathbf{E} : \mathbb{R}^{n\times m} \to \mathbb{R}$ for which the corresponding functionals or the "energy integrals"

$$\mathbb{E}[f] \; := \; \int_\Omega \mathbf{E}(Df)\,\mathrm{d}x \tag{1.1}$$

are lower semicontinuous in appropriate Sobolev spaces, for instance

$$\int_\Omega \mathbf{E}(Df(x)) \leqslant \liminf_{j\to\infty} \int_\Omega \mathbf{E}(Df_j(x)) \tag{1.2}$$

whenever

$$f_j \xrightarrow{w^*} f \; \text{ in } W^{1,\infty}(\Omega, \mathbf{R}^m).$$

For such functionals the problem of minimizing the integral $\mathbb{E}[f]$, under any given (sufficiently regular, e.g. continuous) boundary values always admits a solution, in reasonable domains Ω (e.g. Ω bounded with $\partial\Omega$ smooth). Hence characterisations of such functionals are of fundamental importance in vectorial calculus of variations as well as in its applications in mathematics, physics, engineering and beyond.

In his fundamental work [42], [43] Morrey showed that lower semicontinuity of the functional $\mathbb{E}[f]$ is equivalent to a property which he coined *quasiconvexity*. This asks that for every linear mapping $A:\mathbb{R}^n\to\mathbb{R}^m$ and for every function $f\in A+\mathbb{C}_\circ^\infty(\Omega,\mathbb{R}^m)$ the inequality

$$\mathbb{E}[f] := \int_\Omega \mathbf{E}(Df)\,\mathrm{d}x \geqslant \int_\Omega \mathbf{E}(A)\,\mathrm{d}x = \mathbf{E}(A)|\Omega| \tag{1.3}$$

holds in any given bounded domain $\Omega\subset\mathbb{R}^n$. In other words, the condition requires that compactly supported perturbations of linear maps do not decrease the value of the integral (1.3). However, except in the one dimensional case, it is still an highly open question if quasiconvexity can be described by more explicit conditions.

1.1. **The one dimensional case.** In case either dimension $m=1$ or $n=1$, it is quickly shown that (1.2) and (1.3) are equivalent to the convexity of $\mathbf{E}$. For instance, if $m=n=1$ and $\mathbf{E}$ fails to be convex,

$$\mathbf{E}(c) > t_0\mathbf{E}(a)+(1-t_0)\mathbf{E}(b), \qquad c=t_0a+(1-t_0)b\in[a,b],$$

then set

$$f(x)=\begin{cases} ax, & x\in[0,t_0] \\ b(x-t_0)+at_0, & x\in[t_0,1],\end{cases}$$

and extend $f(x)$ to $\mathbb{R}$ by requiring $f(x+n)=f(x)+nf(1)$. Letting $f_j(x):=f(jx)/j,\ j\in\mathbb{N}$, we have

$$\int_0^1 \mathbf{E}(f_j'(x))\,dx \to t_0\mathbf{E}(a)+(1-t_0)\mathbf{E}(b) < \mathbf{E}(c) = \int_0^1 \mathbf{E}(f_\infty')$$

even if

$$f_j(x):=f(jx)/j \xrightarrow{w^*} f_\infty(x)\equiv cx \text{ in } W^{1,\infty}((0,1),\mathbf{R}).$$

Similarly, quasiconvexity fails since f_j and the linear f_∞ share the same boundary values on the unit interval.

1.2. **Higher dimensions.** On the other hand, in higher dimensions we have examples such as the Jacobian determinant $\mathbf{E}(A)=\det(A)$ which provide non-convex, yet quasiconvex functionals. Indeed, the value of the integral of the Jacobian determinant depends only on the boundary values of the test function, that is

$$\int_\Omega \det(Df)\,\mathrm{d}x = \int_\Omega \det(Dg)\,\mathrm{d}x$$

whenever $f,g\in\mathbb{C}^\infty(\overline{\Omega},\mathbb{R}^n)$ with $f(x)=g(x)$ for $x\in\partial\Omega$.

The natural problem of characterizing quasiconvex functionals in higher dimensions still remains highly open. The are, though, some know conditions sufficient for quasiconvexity. For instance, a functional $\mathbf{E}(A)$ is termed *polyconvex* if it can be represented as a convex function of the minors (subdeterminants) of the matrix A - in particular, the above example of the Jacobian determinant falls in this class. It is known, see [12], that polyconvex functionals satisfy (1.3). Quasiconvexity, though, is a strictly more general condition [28], but proven examples outside polyconvexity are sparse.

A weaker notion is that of *rank-one convexity*. A modification of the one dimensional description, this requires that $t \mapsto \mathbf{E}(A + tX)$ is convex for any fixed matrix A and for any rank one matrix X. Rank-one convexity of an integrand is a concrete local condition (relatively) easy to verify. That quasiconvexity implies rank-one convexity is easy to see, by arguing in a similar fashion as above for the one dimensional case. However, sufficiency of the rank-one convexity remained open for quite some time. The question was settled finally in the negative when Šverák [52] constructed rank-one convex functionals which are not quasiconvex.

On the other hand, the celebrated counter-example of Šverák works only in dimensions $m \geqslant 3$, see [48]. This leaves the possibility for different outcome in dimension 2. For instance, Faraco and Szekelyhidi [30] show that the quasiconvex hull of any compact set of 2×2 matrices can be localized - a fact which fails in higher dimensions, as shown by Kristensen [34].

Morrey himself was not quite definite in which direction he thought things should be true, see [42], [43], and [14, Sect. 9]. We reveal our own thoughts on the matter by recalling the following conjecture in the spirit of Morrey:

Conjecture 1.1. *Rank-one convex functions $\boldsymbol{E} : \mathbb{R}^{2\times 2} \to \mathbb{R}$ are quasiconvex.*

1.3. **The Burkholder functional.** In these lecture notes we propose an approach towards establishing quasiconvexity of functionals in two dimensions, using the associated complex structure. For this, instead of studying general rank-one functionals we concentrate on a specific example, the Burkholder functional from [26].

Actually, for the methods we will apply instead of convexity, it is more appropriate to approach the *concavity*: One says that $\mathbf{E}$ is rank-one concave (resp. quasiconcave) if $-\mathbf{E}$ is rank-one convex (resp. quasiconvex), and null-Lagrangian if both quasiconvex and quasiconcave. The most famous (and, arguably, the most important) rank-one concave function in two dimension is the Burkholder functional from [26], defined for any 2×2 matrix A by

$$\mathbf{B}_p\,(A) = \Big(\,\frac{p}{2}\det A \;+\; (1-\frac{p}{2})\,\big|A\big|^2\,\Big)\cdot |A|^{p-2}, \qquad p \geqslant 2. \tag{1.4}$$

Above, we have chosen the normalization $\mathbf{B}_p\,(Id\,) = 1$ with the identity matrix and the absolute value notation is reserved for the operator norm.

The functional was discovered by Burkholder in a completely different setting, in his studies of optimal estimates for stochastic integrals and martingales, see [26, 24]. For $2 \leqslant p < \infty$ the functional $\mathbf{B}_p\,(A)$ is rank-one concave - for $1 \leqslant p \leqslant 2$ it is rank-one convex. There are [7] also versions of the functional for any $p < 1$, $p \in \mathbb{R}$, and [8] even for complex values of the parameter p.

We also refer the reader to [9] where a multitude of further special properties of the Burkholder functional are discussed.

1.4. **Iwaniec Conjecture.** One of the particularly fascinating features of the Burkholder functional and its conjectured quasiconcavity is the connection of this problem to the famous conjecture of Iwaniec on the p-norm of the *Beurling Transform.* This transform is a Calderón-Zygmund type singular integral,

$$\mathbf{S}\omega \;=\; -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\omega(\xi)\,\mathrm{d}\xi}{(z-\xi)^2}, \tag{1.5}$$

which in many respects plays the role of the Hilbert transform in two dimensions. The Beurling transform is an isometry in $\mathscr{L}^2(\mathbb{C})$, and as a Calderón-Zygmund operator $\mathbf{S}$ is bounded in $\mathscr{L}^p$, but determining here the operator norm is a much harder question. The as yet unsolved conjecture [36] of Iwaniec asserts that

Conjecture 1.2. *For all* $1 < p < \infty$ *it holds*

$$\|\mathbf{S}\|_{\mathscr{L}^p(\mathbb{C})} = p^* - 1 := \begin{cases} p-1\,, & \text{if} \quad 2 \leqslant p < \infty \\ 1/(p-1)\,, & \text{if} \quad 1 < p \leqslant 2 \end{cases} \tag{1.6}$$

The full quasiconcavity of the Burkholder functional $\mathbf{B}_p$ would, among its many potential consequences, imply also (1.6). This follows from another very useful inequality of Burkholder [26]. Namely, with the positive constant $C_p = p\left(1-\frac{1}{p}\right)^{1-p}$ for $p \geqslant 2$, we have

$$C_p \cdot \big(|f_z|^p - (p-1)^p |f_{\bar z}|^p \big) \leqslant \big(|f_z| - (p-1)\, |f_{\bar z}| \big) \cdot \big(|f_z| + |f_{\bar z}| \big)^{p-1} \equiv \mathbf{B}_p(Df)$$

Thus Burkholder's functional can be viewed as an effective rank-one concave majorant of the p-norm functional of the left hand side, see Section 9. It is because of this connection why Morrey's Problem becomes relevant to Conjecture 1.2. For precise statements and further information on related topics we refer to [17, 19, 26, 53].

It is also appropriate to emphasise the origin of the Burkholder's functional in his groundbreaking work on sharp estimates for martingales [23] - [26]. This work has been later on extended in various ways, including applications to computing optimal or almost optimal estimates for norms of singular integrals, e.g. of the Beurling operator. Also the Bellman function techniques (see e.g. [46]) are closely related. We mention only [18], [20], [26], [29], [31], [46], [47], [49], [53] and refer to the recent survey [17] for a wealth of information and an extensive list of references.

2. Integral estimates for the Burkholder functional

Perhaps the main theme of these notes is to describe how with holomorphic deformations one can establish a partial quasiconcavity for the Burkholder functional, in case of non-negative integrands and for perturbations of the identity map:

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and denote by $Id : \Omega \to \mathbb{R}^2$ the identity map. Assume that $f \in Id + \mathbb{C}_\circ^\infty(\Omega)$ satisfies $\boldsymbol{B}_p\big(Df(x)\big) \geqslant 0$ for $x \in \Omega$. Then*

$$\int_\Omega \boldsymbol{B}_p\,(Df)\, dx \leqslant \int_\Omega \boldsymbol{B}_p\,(Id\,)\, dx = |\Omega|, \qquad p \geqslant 2,$$

or, written explicitly

$$\int_\Omega \Big(\frac{p}{2}\, J(z,f) + (1-\frac{p}{2})\, |Df|^2 \Big) \cdot |Df|^{p-2} \leqslant |\Omega|. \tag{2.1}$$

Our proof of the above result is based on holomorphic deformations and quasiconformal methods. In turn, the result has a multitude of implications e.g. towards optimal distortion bounds for quasiconformal mappings or sharp integral estimates for the Beurling transform.

From the point of view of the theory of nonlinear elasticity of Antman [3], Ball [12, 16], Ciarlet [27] and many others, for homogeneous materials the elastic deformations $f\colon \Omega \to \mathbb{R}^n$ are minimizers of a given energy integral

$$\mathscr{E}[f] \;=\; \int_\Omega \mathbf{E}(Df)\,\mathrm{d}x \;<\infty \tag{2.2}$$

where the so-called *stored energy function* $\mathbf{E}\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ carries the mechanical properties of the elastic material in Ω. By virtue of the principle of non-interpenetration of matter the minimizers ought to be injective. It is from these perspectives that our energy-estimates, although limited to (quasiconformal) homeomorphisms, are certainly not short of applications.

As one might expect, passing to the limit in (2.1) as $p \to 2$ or as $p \to \infty$ will yield interesting sharp inequalities. The first mentioned limit leads to

Corollary 2.2. *Given a bounded domain $\Omega \subset \mathbb{R}^2$ and a homeomorphism $f\,:\,\Omega \xrightarrow{\text{onto}} \Omega$ such that*

$$f(z) - z \in \mathscr{W}^{1,2}_0(\Omega),$$

we then have

$$\int_\Omega \big(1 \,+\, \log|Df(z)|^2\,\big)\; J(z,f)\,\mathrm{d}z \;\;\leqslant\;\; \int_\Omega |Df(z)|^2\,\mathrm{d}z \tag{2.3}$$

Equality occurs for the identity map, as well as for a number of piece-wise radial mappings discussed in Section 8.

Reflecting back on Conjecture 1.1, note that the functional

$$\mathscr{F}(A) = \big(1 \,+\, \log|A|^2\,\big)\;\det(A) \;-\; |A|^2$$

is rank-one concave. However, with growth stronger than quadratic it is not polyconcave [12], i.e. cannot be written as a concave function of the minors of A. According to Corollary 2.2 this functional is nevertheless quasiconcave with respect to homeomorphic perturbations of the identity. Going to the inverse maps yields a quasiconvexity result for the functional

$$\mathscr{H}(A) := \frac{1}{2}\frac{|A|^2}{\det A} + \log\big(\det A\big) - \log|A|, \qquad\qquad \det A > 0. \tag{2.4}$$

This can be interpreted as a sharp integrability of $\log J(z,f)$ for planar maps of finite distortion, see [7].

It is of course classical [44, 32] that the nonlinear differential expression $J(z,f)\log|Df(z)|^2$ belongs to $\mathscr{L}^1_{\mathrm{loc}}(\Omega)$ for every $f\in\mathscr{W}^{1,2}_{\mathrm{loc}}(\Omega)$ whose Jacobian determinant $J(z,f)=\det Df(z)$ is nonnegative. The novelty in (2.3) lies in the best constant $C=1$ in the right hand side, and the proof of $\mathscr{L}\log\mathscr{L}$-integrability of the Jacobian is new.

In turn, the limit $p\to\infty$ yields the following sharp inequality.

Corollary 2.3. *Denote by* $\mathbf{S}$ *the Beurling-Ahlfors operator (defined in (1.5)) and assume that* μ *is a measurable function with* $|\mu(z)|\leqslant\chi_{\mathbb{D}}(z)$ *for every* $z\in\mathbb{C}$*. Then*

$$\int_{\mathbb{D}}\big(1-|\mu(z)|\big)\,e^{|\mu(z)|}\,\big|\exp(\mathbf{S}\mu(z))\big|\,dz\;\leqslant\;\pi. \tag{2.5}$$

Equality occurs for an extensive class of piece-wise radial mappings discussed in Section 8.2.

Prior to the above result, it was known that the area distortion results [4] yield the exponential integrability of $\operatorname{Re}\mathbf{S}\mu$ under the strict inequality $\|\mu\|_\infty<1$, see [5, p. 387].

The proofs of the Corollaries are found in Section 7. Sections 8 and 9 contain further observations on the Burkholder functionals. For example, their local maxima are discussed.

We then shortly describe the ideas behind the proofs of our main results, which are given in Section 6. In fact we will prove a slightly generalized form of Theorem 2.1, where we relax the identity boundary conditions to asymptotic normalization at infinity. This is done in Theorem 6.4 below, where we will interpolate between the natural end-point cases $p=2$ and $p=\infty$. The holomorphic interpolation method used is inspired by the variational principle of thermodynamical formalism and the underlying analytic dependence coming from holomorphic motions. Here these are developed to a key ingredient of our argument, a new variant of the celebrated Riesz-Thorin interpolation theorem.

3. Interpolation

In order to describe the required interpolation result, let $\mathscr{M}(\Omega,\sigma)$ denote the class of complex-valued σ-measurable functions on a measure space

(Ω, σ). The Lebesgue spaces $\mathscr{L}^p(\Omega, \sigma)$ are (quasi-)normed by

$$\|\Phi\|_p = \left(\int_\Omega |\Phi(z)|^p \, \mathrm{d}\sigma(z) \right)^{\frac{1}{p}}, \quad 0 < p < \infty , \quad \text{and } \|\Phi\|_\infty = \operatorname*{ess\,sup}_{z \in \Omega} |\Phi(z)|$$

Let $U \subset \mathbb{C}$ be a domain. We shall consider analytic families f_λ of measurable functions in Ω, i.e. jointly measurable functions $(x, \lambda) \mapsto f_\lambda(x)$ defined on $\Omega \times U$ such that for each fixed $x \in \Omega$ the map $\lambda \to f(x, \lambda)$ is analytic in U. The family is said to be *non-vanishing* if there exists a set $E \subset \Omega$ of σ-measure zero such

$$f(x, \lambda) \neq 0 \qquad \text{for all } x \in \Omega \setminus E \text{ and } \lambda \in U. \tag{3.1}$$

We state our interpolation result first in the setting of the right half plane, $U = \mathbb{H}_+ := \{\lambda : \operatorname{Re} \lambda > 0\}$, in order to facilitate comparison with the Riesz-Thorin theorem:

Lemma 3.1 (Interpolation Lemma). *Let $0 < p_0, p_1 \leqslant \infty$ and let $\{\Phi_\lambda ; \lambda \in \mathbb{H}_+\} \subset \mathscr{M}(\Omega, \sigma)$ be an analytic and non-vanishing family, with complex parameter λ in the right half plane. Assume further that for some $a \geqslant 0$,*

$$M_1 := \| \Phi_1 \|_{p_1} < \infty \ \text{ and } \ M_0 := \sup_{\lambda \in \mathbb{H}_+} e^{-a \operatorname{Re} \lambda} \| \Phi_\lambda \|_{p_0} < \infty.$$

Then, letting $M_\theta := \left\| \Phi_\theta \right\|_{p_\theta}$ with $\dfrac{1}{p_\theta} = (1-\theta) \cdot \dfrac{1}{p_0} + \theta \cdot \dfrac{1}{p_1}$, we have for every $0 < \theta < 1$,

$$M_\theta \leqslant M_0^{1-\theta} \cdot M_1^\theta < \infty \tag{3.2}$$

Remark 3.2. Compared to Riesz-Thorin, our result needs the bound for the other end-point exponent only at one single point, when $\lambda = 1$! However, without the non-vanishing condition the conclusion of the interpolation lemma breaks down drastically. A simple example (where $a = 0$) is obtained by taking $p_0 = 1$, $p_1 = \infty$, and considering the family $f(x, \lambda) = \left(\frac{1-\lambda}{1+\lambda}\right) g(x)$ for $\operatorname{Re} \lambda > 0$ and $x \in \Omega$, for a function $g \in \mathscr{L}^1(\Omega, \sigma) \setminus \left(\bigcup_{p>1} \mathscr{L}^p(\Omega, \sigma) \right)$.

In applications one often has rotational symmetry, thus requiring a unit disk version of the interpolation. After a Möbius transform in the parameter plane the interpolation lemma runs as follows (observe that we have interchanged the roles of the indices p_0 and p_1 only for aesthetic reasons):

Lemma 3.3 (Interpolation Lemma for the disk)**.** *Let* $0 < p_0, p_1 \leqslant \infty$ *and* $\{\Phi_\lambda\,;\ |\lambda| < 1\} \subset \mathscr{M}(\Omega, \sigma)$ *be an analytic and non-vanishing family with complex parameter* λ *in the unit disc. Suppose*

$$M_0 := \|\,\Phi_0\|_{p_0} < \infty, \quad M_1 := \sup_{|\lambda|<1} \|\,\Phi_\lambda\|_{p_1} < \infty \quad \textit{and} \quad M_r := \sup_{|\lambda|=r} \big\|\Phi_\lambda\big\|_{p_r},$$

where

$$\frac{1}{p_r} = \frac{1-r}{1+r}\cdot\frac{1}{p_0} + \frac{2r}{1+r}\cdot\frac{1}{p_1}$$

Then, for every $0 \leqslant r < 1$ *, we have*

$$M_r \leqslant M_0^{\frac{1-r}{1+r}} \cdot M_1^{\frac{2\,r}{1+r}} < \infty \tag{3.3}$$

Before embarking into the proof of the interpolation Lemma, let us remark that often analytic families of functions are defined by considering analytic functions having values in the Banach space $\mathscr{L}^p(\Omega, \sigma)$ for $p \geqslant 1$. In this case it is well-known (e.g. [50, Thm. 3.31]) that one may define analyticity of the family by several equivalent conditions, e.g. by testing elements from the dual. This notion agrees with the definition given in the introduction, see [5, Lemma 5.7.1].

Proof of Lemma 3.1. We may, and do, assume that $M_0 = 1$ and that $a = 0$; the case $a > 0$ reduces to this by simply considering the analytic family $e^{-a\lambda}\Phi_\lambda(x)$. Similarly by taking restrictions we may assume $\sigma(\Omega) < \infty$.

We first consider the case $0 < p_0, p_1 < \infty$, and establish the result in the situation where for a fixed $A \in (1, \infty)$ there is the uniform bound

$$\frac{1}{A} \leqslant |\Phi_\lambda(x)| \leqslant A \qquad \text{for all} \quad \lambda \in \mathbb{H}_+ \ \text{ and } \ x \in \Omega. \tag{3.4}$$

This is to ensure that all of our integrals and computations below are meaningful. At the end of the proof we get rid of this extra assumption.

Let $\theta \in (0, 1)$ be given as in the statement of the lemma. First, we will find the support line to the convex function $\frac{1}{p} \mapsto \log \|\Phi_\theta\|_p$ at $\frac{1}{p_\theta}$. We are looking for a function $u_p(\theta)$ with the following properties,

$$u_p(\theta) = \frac{1}{p}\, I + u_\infty(\theta) \leqslant \log \|\Phi_\theta\|_p \quad \text{and} \quad u_{p_\theta}(\theta) = \log \|\Phi_\theta\|_{p_\theta}, \tag{3.5}$$

where I and $u_\infty(\theta)$ are independent of p. Using the concavity of the logarithm function we can write down these terms explicitly. Indeed, by concavity, for any probability density $\wp(x)$ on Ω and for any exponent $0 < p < \infty$,

$$\frac{1}{p}\int_\Omega \wp(x)\log\left(\frac{|\Phi_\theta(x)|^p}{\wp(x)}\right)\,\mathrm{d}\sigma \leqslant \log\|\Phi_\theta\|_p,$$

where equality holds for $p = p_\theta$ with the following choice of density

$$\wp(x) \;:=\; \frac{\big|\Phi_\theta(x)\big|^{p_\theta}}{\int_\Omega \big|\Phi_\theta(y)\big|^{p_\theta}\,\mathrm{d}\sigma(y)}, \quad \int_\Omega \wp(x)\,\mathrm{d}\sigma(x) \;=1. \tag{3.6}$$

It is useful to note that because of our assumptions (3.4), the $\wp$ is uniformly bounded from above and below. With this in mind we find the coefficients in (3.5), by using the fixed density (3.6) and by writing

$$I := \int_\Omega \wp(x)\log\left(\frac{1}{\wp(x)}\right)\mathrm{d}\sigma \quad \text{and} \quad u_\infty(\theta) := \int_\Omega \wp(x)\log|\Phi_\theta(x)|\,\mathrm{d}\sigma.$$

The key idea in this representation is that we may embed the line $u_p(\theta)$ in a harmonic family of lines parametrized by $\lambda \in \mathbb{H}_+$,

$$u_p(\lambda) := \frac{1}{p}\,I + u_\infty(\lambda) \;=\; \frac{1}{p}\int_\Omega \wp(x)\log\left(\frac{|\Phi_\lambda(x)|^p}{\wp(x)}\right)\,\mathrm{d}\sigma.$$

It is important to notice that we kept the slope I fixed and because of the non-vanishing assumption the constant term $u_\infty(\lambda)$ becomes a harmonic function of λ. Again, in view of Jensen's inequality we have the envelope property, the analogue of (3.5) for all $\lambda \in \mathbb{H}_+$ and $0 < p \leqslant \infty$,

$$u_p(\lambda) \leqslant \log\|\Phi_\lambda\|_p \quad \text{and} \quad u_{p_\theta}(\theta) = \log\|\Phi_\theta\|_{p_\theta}. \tag{3.7}$$

By our assumptions, for $p = p_0$ we thus have $u_{p_0}(\lambda) \leqslant \log\|\Phi_\lambda\|_{p_0} \leqslant 0$ for all $\lambda \in \mathbb{H}_+$. Here Harnack's inequality for nonpositive harmonic functions in $\mathbb{H}_+$ takes a particularly simple form when restricted to the interval $\theta \in (0,1)$:

$$u_{p_0}(\theta) \leqslant \theta\, u_{p_0}(1) \qquad \text{for } \theta \in (0,1). \tag{3.8}$$

Finally combining the estimates (3.7) and (3.8) yields

$$\begin{aligned}
\log\|\Phi_\theta\|_{p_\theta} \;&=\; u_{p_\theta}(\theta) = u_{p_0}(\theta) + \left(\frac{1}{p_\theta} - \frac{1}{p_0}\right) I \\
&\leqslant\; \theta\, u_{p_0}(1) + \theta\left(\frac{1}{p_1} - \frac{1}{p_0}\right) I \\
&=\; \theta\, u_{p_1}(1) \leqslant \theta\,\log M_1,
\end{aligned}$$

which is exactly what we aimed to prove.

The argument can easily be adapted to accommodate the cases when $p_0 = \infty$ or $p_1 = \infty$. We will instead use a limiting argument. First, normalize to $\sigma(\Omega) = 1$, then $\|\cdot\|_p$ increases with p and one has $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$. Hence, as (3.4) holds, we obtain the desired result by approximating the possibly infinite exponents by finite ones.

Let us finally dispense with the extra assumption (3.4). Since the removal of a null set from Ω is allowed, we may assume that the non-vanishing condition holds for every $x \in \Omega$, i.e. one may take $E = \emptyset$ in (3.1). Choose first a family $\varphi_n(\lambda)$ of Möbius transformations such that

$$\varphi_n(1) = 1, \qquad \lim_{n\to\infty} \varphi_n(\lambda) = \lambda, \;\; \lambda \in \mathbb{H}_+, \qquad \text{and} \quad \overline{\varphi_n(\mathbb{H}_+)} \subset \mathbb{H}_+, \;\; n \in \mathbb{N},$$

and let for any positive integer k,

$$\Omega_{n,k} := \{x \in \Omega \;:\; |\Phi_\lambda(x)| \in [1/k, k] \;\text{ for all }\; \lambda \in \overline{\varphi_n(\mathbb{H}_+)}\;\}. \tag{3.9}$$

The measurable sets $\Omega_{n,k}$ fill the space, $\bigcup_{k=1}^\infty \Omega_{n,k} = \Omega$. Moreover, for each fixed integer $k \geqslant 1$ the non-vanishing analytic family

$$(x, \lambda) \mapsto \;\Phi_{\varphi_n(\lambda)}(x), \qquad x \in \Omega_{n,k}, \;\; \lambda \in \mathbb{H}_+,$$

satisfies the uniform bound (3.4), and thus we may interpolate it. Letting $k \to \infty$ gives $\|\Phi_{\varphi_n(\theta)}\|_{p_\theta} \leqslant M_1^\theta$, and the claim (3.2) follows by Fatou's lemma taking a second limit $n \to \infty$. □

We remark that the Harnack inequality used above can be deduced from the standard Harnack inequality for negative harmonic functions in the unit disc, $u(w) \leqslant \frac{1-|w|}{1+|w|}\, u(0)$, by a change of variables $\lambda = (1-w)/(1+w)$. The very same change of variables allows one to deduce Lemma 3.3 for the values $\lambda \in (0,1)$ as a consequence of Lemma 3.1, and rest follows from rotational symmetry.

4. Quasiconformal distortion

We start with some preliminaries. Our goal is to apply the Interpolation Lemma 3.3 in estimating the variational integrals such as (2.1), and therefore we look for analytic and nonvanishing families of gradients of mappings.

In view of the *Lambda-lemma* [41], to be discussed in Section 5, this takes us to the notion of quasiconformal mappings. By definition, in any dimension $n \geqslant 2$ these are homeomorphisms $f : \Omega \to \Omega'$ in the Sobolev class $\mathscr{W}^{1,n}_{loc}(\Omega)$ for which the differential matrix $Df(x) \in \mathbb{R}^{n\times n}$ and its determinant are coupled in the *distortion inequality*,

$$|Df(x)|^n \leqslant K(x) \det Df(x) \ , \quad \text{where} \quad |Df(x)| = \max_{|\xi|=1} |Df(x)\xi|, \tag{4.1}$$

for some bounded function $K(x)$. The smallest $K(x) \geqslant 1$ for which (4.1) holds almost everywhere is referred to as the *distortion function* of the mapping f. We call f K-quasiconformal if $K(f) := \|K(x)\|_\infty \leqslant K$.

In dimension $n = 2$ it is useful to employ complex notation by introducing the Cauchy-Riemann operators

$$\partial f = f_z = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad \text{and} \quad \bar\partial f = f_{\bar z} = \frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

Writing

$$|Df(z)| = |f_z| + |f_{\bar z}| \qquad \text{and} \qquad \det Df(z) = J(z,f) = |f_z|^2 - |f_{\bar z}|^2,$$

we see that for a planar Sobolev homeomorphism f the K-quasiconformality simplifies to a linear equation

$$f_{\bar z} = \mu(z) f_z \tag{4.2}$$

called the *Beltrami equation*. Here the *dilatation function* μ is measurable and satisfies $\|\mu\|_\infty =: k = \frac{K-1}{K+1} < 1$.

The Beltrami equation will then enable holomorphic deformations of the homeomorphism f and of its gradient. Indeed, under a proper normalization the solutions to (4.2) and their derivatives depend holomorphically on the coefficient μ, see [5, p. 188].

For choosing the normalization, recall that Theorem 2.1 considers the identity boundary values, and thus mappings that extend conformally outside Ω. We therefore look for solutions to (4.2) defined in the entire plane $\mathbb{C}$, with the dilatation μ vanishing outside the domain Ω. On the other hand, the identity boundary values cannot be retained under general holomorphic deformations; one needs to content with the asymptotic normalization

$$f(z) = z + b_1 z^{-1} + b_2 z^{-2} + \cdots \ , \text{ for } |z| \to \infty \tag{4.3}$$

Global $\mathscr{W}^{1,2}_{loc}$-solutions to (4.2) with these asymptotics are called *principal solutions*. They exist and are unique for each coefficient μ supported in the bounded domain Ω, and each of them is a homeomorphism. They can be found simply in the form of the Cauchy transform

$$f(z) = z + \frac{1}{\pi}\int_{\mathbb{C}} \frac{\omega(\xi)\,\mathrm{d}\xi}{z-\xi}\ , \quad \text{where } \omega = f_{\bar z} \in \mathscr{L}^2(\mathbb{C}) \tag{4.4}$$

Substituting $\omega := f_{\bar z}$ into (4.2) yields a singular integral equation for the unknown density function $\omega \in \mathscr{L}^2(\mathbb{C})$,

$$\omega \ - \ \mu\,\mathbf{S}\omega \ = \ \mu \ \in \mathscr{L}^2(\mathbb{C}), \tag{4.5}$$

where the Beurling Transform is an isometry in $\mathscr{L}^2(\mathbb{C})$. Whence (4.5) can be solved by the Neumann series

$$\omega = \mu + \mu\,\mathbf{S}\mu + \mu\,\mathbf{S}\mu\,\mathbf{S}\mu + \cdots \tag{4.6}$$

converging in $\mathscr{L}^2(\mathbb{C})$. In particular, we see that f and its derivatives $f_z, f_{\bar z}$ depend holomorphically on μ !

We refer to the well-known monographs [1], [5] and [35] for the basic properties and further details on quasiconformal mappings.

Returning to the integral estimates of Theorem 2.1, one observes that the condition $\mathbf{B}_p\,(Df) \geqslant 0$ in the Theorem is equivalent to $|Df|^2 \leqslant \frac{p}{p-2}J_f$, which actually amounts to quasiconformality of f. Indeed, in this setting our result reads as follows:

Theorem 4.1. *Let $f : \Omega \longrightarrow \Omega$ be a K-quasiconformal map of a bounded open set $\Omega \subset \mathbb{C}$ onto itself, extending continuously up to the boundary, where it coincides with the identity map $Id(z) \equiv z$. Then*

$$\int_\Omega \boldsymbol{B}_p\,(Df)\,dx \ \leqslant \int_\Omega \boldsymbol{B}_p\,(Id\,)\,dx \ = \ |\Omega|, \quad \text{for all} \quad 2 \leqslant p \leqslant \frac{2K}{K-1}\,.$$

Further, the equality occurs for a class of (expanding) piecewise radial mappings discussed in Section 8.

This result says, roughly, that the Burkholder functional is quasiconcave within quasiconformal perturbations of the identity. It is quite interesting that there is an equality in the above theorem for a large class of radial-like maps. When smooth and $p < 2K/(K-1)$ these are all local maxima for the functional, see Proposition 8.2 and Corollary 8.3 for details. In particular, the identity map is a *local maximum* of all the Burkholder functionals in (1.4).

Among the strong consequences of Theorem 4.1, one obtains (with the same assumptions as in Theorem 4.1) that

$$\frac{1}{|\Omega|}\int_\Omega \big|Df(z)\big|^p \,\mathrm{d}z \;\leqslant\; \frac{2K}{2K \;-\; p\,(K-1)}\,, \quad for\;\; 2 \leqslant p < \frac{2K}{K-1} \tag{4.7}$$

with equality for piecewise power mappings, such as $f(z) = |z|^{1-1/K}z$ in the unit disk, see Corollary 7.1 below. The $\mathscr{W}^{1,p}$-regularity of $K-$quasiconformal mappings, for $p < 2K/(K-1)$, was established in [4], as a corollary of the area distortion theorem. However, there the bounds for integrals such as in (4.7) were described in terms of unspecified constants depending on the distortion bound K. Here we have obtained the sharp explicit bound for the $\mathscr{L}^p$-integrals of the derivatives of K-quasiconformal mappings.

Similarly, for any K-quasiregular mapping $f \in \mathscr{W}^{1,2}_{loc}(\Omega)$, injective or not, we can improve the local $\mathscr{W}^{1,p}$-regularity to weighted integral bounds at the borderline exponent $p = 2K/(K-1)$,

$$\left(\frac{1}{K(x)} \;-\; \frac{1}{K}\right)\, \big|Df(x)\big|^{\frac{2K}{K-1}} \in \mathscr{L}^1_{loc}(\Omega). \tag{4.8}$$

We refer to Section 6 for a more thorough discussion and Section 8 for elaborate examples of extremal mappings.

5. Holomorphic deformations

The computer animations revolutionized the complex dynamics in the early 80's, and among phenomena observed was strong geometric - not only topological - stability in perturbations of hyperbolic systems; see Figure 1 for a typical illustration. Mañé, Sad and Sullivan [41] realized that these phenomena can be completely understood in terms of the following fundamental notion, which will intimately tie up the holomorphic deformations with the quasiconformal mappings, see Theorem 5.2 below.

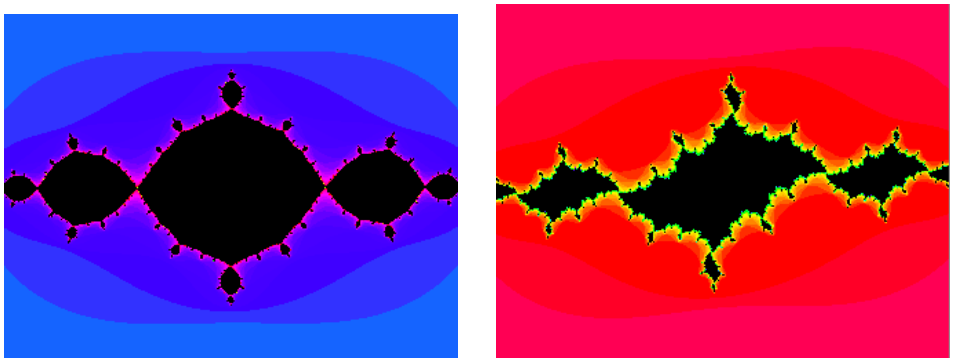

Figure 1. Two Julia sets with parameters from the same component of the Mandelbrot set.

Definition 5.1. Let A be any subset of the complex plane $\mathbb{C}$. Then a holomorphic motion of A, parametrized by the unit disk $\mathbb{D}$, is a map

$$\Phi : \mathbb{D} \times A \to \mathbb{C}$$

such that

i) For any fixed $\lambda \in \mathbb{D}$, the map

$$a \to \Phi(\lambda, a) = \Phi_\lambda(a) \quad \text{is an injection.}$$

ii) For any fixed $a \in A$, the map

$$\lambda \to \Phi(\lambda, a) \text{ is holomorphic in } \mathbb{D}.$$

iii) The mapping Φ_0 is the identity on A,

$$\Phi(0, a) = a, \quad \text{for every } a \in A, \text{ and}$$

Thus the holomorphic motions represent minimal assumptions for a notion of isotopy of a set A, where the dependence on time λ is holomorphic. However, the Mañé-Sad-Sullivan Lambda-lemma [41] shows that necessarily each Φ_λ is a quasisymmetric mapping of A, with quasisymmetry constants depending only on $|\lambda|$. In particular, Φ extends to a motion of the closure $\overline{A}$. This, with basic results from complex dynamics, already explains the geometric rigidity observed in pictures such as Figure 1. On the other hand, in case we are considering holomorphic deformations of the whole space, since quasisymmetric maps of $\mathbb{C}$ are quasiconformal, a closer look at Mañé-Sad-Sullivan Lambda-lemma leads to the following result.

Theorem 5.2. *The following conditions are equivalent:*

i) $f_\lambda(z) = \Phi(\lambda, z)$ *defines a holomorphic motion of* $\mathbb{C}$*, fixing the points* 0, 1.

ii) $f_\lambda \in W^{1,1}_{loc}(\mathbb{C})$ *are homeomorphic solutions to the PDE*

$$(*) \qquad \partial_{\bar z} f(z) = \mu_\lambda(z) \partial_z f(z), \qquad f(0) = 0, f(1) = 1$$

where $\lambda \mapsto \mu_\lambda(z)$ *holomorphic, with* $\|\mu_\lambda\|_\infty \leqslant |\lambda|, \quad \lambda \in \mathbb{D}$.

For further details see [5, Chapter 12]. Instead of maps normalised at 0 and 1, one may also consider principal solutions.

Conversely, given any quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$, which is a principal solution to (4.2), take the homeomorphic principal solutions $f_\lambda(z)$ to (4.2) for the coefficients $\mu_\lambda(z) := (\lambda/k)\mu(z)$. Then by (4.4)-(4.6) the maps f_λ depend holomorphically on the parameter $\lambda \in \mathbb{D}$, in fact, they define a holomorphic motion with $\Phi_k = f$ the original map $f : \mathbb{C} \to \mathbb{C}$. Hence any quasiconformal mapping can be embedded to a holomorphic motion.

In fact, even stronger conclusions are possible with help of the generalised Lambda-lemma of Slodkowski [51]. This shows that any holomorphic motion of any set A extends to a motion of the whole space $\mathbb{C}$. In any case, the Lambda-lemmas explain why holomorphic deformations and quasiconformal estimates are so intimately tied together.

Moreover, it useful to observe that actually for all analytic families of quasiconformal maps, the derivatives $\partial_z f_\lambda$ are non-vanishing as required by the interpolation Lemmas 3.1-3.3, i.e. non-vanishing outside a common set of measure zero. For details see [7, Remark 3.6]

6. Proofs of the Main Theorems

We next recall some standard facts on smooth approximation of principal solutions of the Beltrami equation, for details see [7]. First, we have

Lemma 6.1 ($\mathscr{C}^{1,\alpha}$-regularity)**.** *The principal solution of the Beltrami equation (4.2) in which* $\mu \in \mathscr{C}^\alpha(\mathbb{C})$, $0 < \alpha < 1$, *is a* $\mathscr{C}^{1,\alpha}(\mathbb{C})$ *- diffeomorphism. In particular,* $|f_z|^2 \geqslant J(z, f) > 0$, *everywhere.*

As might be expected, almost everywhere convergence of the Beltrami coefficients yields $\mathscr{W}^{1,2}_{\mathrm{loc}}$- convergence of the principal solutions. The precise statement reads as follows:

Lemma 6.2 (Smooth Approximation)**.** *Suppose the Beltrami coefficients* $\mu_\ell \in \mathscr{C}_\circ^\infty(\Omega)$ *satisfy* $|\mu_\ell(z)| \leqslant k < 1$ *, for all* $\ell = 1, 2, ...,$ *and converge almost everywhere to* μ *. Then the associated principal solutions* $f^\ell : \mathbb{C} \to \mathbb{C}$ *are* $\mathscr{C}^\infty$*-smooth diffeomorphisms converging in* $\mathscr{W}^{1,2}_{\mathrm{loc}}(\mathbb{C})$ *to the principal solution of the limit equation* $f_{\bar z} = \mu(z) f_z$ *.*

Every measurable Beltrami coefficient satisfying $|\mu(z)| \leqslant k\,\chi_\Omega(z)$, $0 \leqslant k < 1$, can be approximated this way.

As the last of the preliminaries, in applying the Interpolation Lemma 3.3 we will need sharp $\mathscr{L}^2$-estimates of gradients, valid for all complex deformations of a given mapping. For the principal solutions in the unit disk $\mathbb{D}$, these result from the classical Area Theorem (see e.g.[5, p. 41]).

Lemma 6.3 (Area Inequality)**.** *The area of the image of the unit disk under a principal solution in* $\mathbb{D}$ *does not exceed* π*. It equals* π *if and only if the solution is the identity map outside the disk.*

Let us recall a proof emphasizing the null-Lagrangian property of the Jacobian determinant. On the circle we have the equality $f(z) = g(z)$, where $g \in \mathscr{W}^{1,2}(\mathbb{D})$ is given by $g(z) = z + \sum\limits_{n\geqslant 1} b_n\, \bar z^{\,n}$, for $|z| \leqslant 1$. This yields

$$\int_{\mathbb{D}} J(z,f)\,\mathrm{d}z = \int_{\mathbb{D}} J(z,g)\,\mathrm{d}z = \int_{\mathbb{D}} \left(1 - |g_{\bar z}(z)|^2\right)\mathrm{d}z \leqslant \pi$$

Equality occurs if and only if $g_{\bar z} \equiv 0$, meaning that all the coefficients b_n vanish.

Having disposed of these lemmas, we can now proceed to the proof of the main integral estimate, where in the complex notation the Burkholder functional takes the form

$$\mathscr{B}^p_\Omega[f] := \int_\Omega \left(|f_z| - (p-1)\,|f_{\bar z}| \right) \cdot \left(|f_z| + |f_{\bar z}| \right)^{p-1} \mathrm{d}z\,, \quad p \geqslant 2. \tag{6.1}$$

We will actually deduce (2.1) from a slightly more general result, where we relax the identity boundary values and allow principal mappings:

Theorem 6.4 (Sharp $\mathscr{L}^p$-inequality)**.** *Let* $f : \mathbb{C} \to \mathbb{C}$ *be the principal solution of a Beltrami equation;*

$$f_{\bar z}(z) = \mu(z)\, f_z(z)\,, \qquad |\mu(z)| \leqslant k\,\chi_{\mathbb{D}}(z)\,, \quad 0 \leqslant k < 1, \tag{6.2}$$

in particular, conformal outside the unit disk $\mathbb{D}$*.*

Then, for all exponents $2 \leqslant p \leqslant 1 + 1/k$*, we have*

$$\int_{\mathbb{D}} \left(1 \; - \; \frac{p\,|\mu(z)|}{1 + \,|\mu(z)|}\right) \; \big(|f_z(z)| + |f_{\bar z}(z)|\big)^p \, \mathrm{d}z \;\; \leqslant \; \pi. \tag{6.3}$$

Equality occurs for some fairly general piecewise radial mappings discussed in Section 8.

The above form of the main result gives a flexible and remarkably precise local description of the $\mathscr{L}^p$-properties of derivatives of a quasiconformal map, especially interesting in the borderline situation $p = 1 + 1/k$. Indeed, combined with the Stoilow factorization, the theorem gives for any $\mathscr{W}^{1,2}_{loc}(\Omega)$-solution to (6.2), injective or not, the estimate

$$\big(k - \,|\mu(z)|\big) \, \big|Df(z)\big|^{1+1/k} \; \in \; \mathscr{L}^1_{loc}(\Omega). \tag{6.4}$$

Thus for all K-quasiregular mappings we obtain optimal weighted higher integrability bounds at the borderline case $p = 2K/(K-1)$. For p below the borderline, the $\mathscr{W}^{1,p}_{loc}$-regularity was established already in [4]. The borderline integrability was previously covered [10] only in the very special case $|\mu| = k \cdot \chi_E$, for $E \subset \mathbb{D}$, in Theorem 6.4.

The proof of Theorem 6.4 applies the Interpolation lemma in conjunction with analytic families of quasiconformal maps. However, the choice of the specific analytic family for our situation is quite non-trivial, in order to enable sharp estimates. In a sense the speed of the change with respect to the analytic parameter must be localized in a delicate manner, see (6.7) below.

Proof of Theorem 6.4. Given a principal solution f to (6.2), with $\mu(z) \equiv 0$ for $|z| > 1$, we are to prove the integral bounds (6.3). There is no loss of generality in assuming that $\mu \in \mathscr{C}^\infty_\circ(\mathbb{D})$, for if not, we approximate μ with $\mathscr{C}^\infty_\circ$-smooth Beltrami coefficients, and, thanks to Fatou's lemma, there is no difficulty in passing to the limit in (6.3). On the other hand, this reduction could be avoided by using the argument of [7, Remark 3.6].

With this assumption we fix an exponent $2 \leqslant p \leqslant 1 + \|\mu\|^{-1}_\infty$ and look for holomorphic deformations of the given function f, via an analytic family of

Beltrami equations together with their principal solutions,

$$F^\lambda_{\bar z} = \mu_\lambda(z)\, F^\lambda_z, \qquad \mu_\lambda(z) = \tau_\lambda(z)\cdot \frac{\mu(z)}{|\mu(z)|} \tag{6.5}$$

Here $\tau_\lambda(z)$ is an analytic function in λ to be chosen later with $|\tau_\lambda(z)| < 1$. We aim to explore Interpolation in the disk, Lemma 3.3, by applying it to a suitable non-vanishing analytic family constructed from the derivatives of $F^\lambda(z)$. Hence the question is the right choice of τ_λ.

We want $F^0(z) \equiv z$, thus $\tau_0(z) \equiv 0$, while for some value $\lambda = \lambda_\circ$ we need to have $\tau_{\lambda_\circ}(z) = |\mu(z)|$, so that $f = F^{\lambda_\circ}$. Comparing the exponents in (6.3) and in Lemma 3.3 suggests that we choose

$$p = 1 + \frac{1}{\lambda_\circ}, \qquad p_0 = \infty, \qquad p_1 = 2. \tag{6.6}$$

These conditions will then be confronted with the need of weighted $\mathscr{L}^2$-bounds consistent with the inequality (6.3).

To make the long story short, we choose

$$\mu_\lambda(z) = \tau_\lambda(z)\cdot\frac{\mu(z)}{|\mu(z)|}, \quad \text{where } \frac{\tau_\lambda(z)}{1+\tau_\lambda(z)} = p\cdot\frac{|\mu(z)|}{1+|\mu(z)|}\cdot\frac{\lambda}{1+\lambda}, \tag{6.7}$$

or more explicitly,

$$\mu_\lambda(z) \;=\; \frac{p\,\lambda\,\mu(z)}{(1+\lambda)\,(1\,+\,|\,\mu(z)|\,)\;-\;p\,\lambda\,|\,\mu(z)|}$$

The complex parameter λ runs over the unit disk, $|\lambda| < 1$. One may visualize $\lambda \mapsto \tau_\lambda(z)$ as the conformal mapping from the unit disk onto the horocycle

$$\left\{ w \in \mathbb{D} :\; 2\ \mathrm{Re}\left(\frac{w}{1+w}\right) < p\cdot\frac{|\mu(z)|}{1+|\mu(z)|}\right\}$$

determined by the weight function in (6.3).

From (6.7) one readily sees that $|\mu_\lambda(z)| \leqslant |\lambda|\,\chi_{\mathbb{D}}(z)$, furthermore $\mu_\lambda \in \mathscr{C}^\alpha(\mathbb{C})$, $0 < \alpha \leqslant 1$. Therefore the equation (6.5) admits a unique principal solution $F^\lambda : \mathbb{C} \to \mathbb{C}$, which is a $\mathscr{C}^{1,\alpha}$ - diffeomorphism. It depends analytically [2] on the parameter λ, as seen by developing (4.5) in a Neumann series, and we have

$$|F^\lambda_z|^2 \;\geqslant\; |F^\lambda_z|^2 \,-\, |F^\lambda_{\bar z}|^2 \;=\; J(z, F^\lambda) \;>\; 0\,, \qquad \text{everywhere in } \mathbb{C}.$$

Moreover, $F^0(z) = z$ with $F^{\lambda_\circ} = f$, where $\lambda_\circ$ was defined by (6.6).

As the non-vanishing analytic family $\{\Phi_\lambda\}_{|\lambda|<1}$ we choose

$$\Phi_\lambda(z) = F^\lambda_z(z)(1+\tau_\lambda(z)). \tag{6.8}$$

Explicitly,

$$\Phi_{\lambda}(z) \;=\; \frac{(1+\lambda)\,(1+|\mu(z)|\,)\;F_z^{\lambda}(z)}{(1+\lambda)\,(1\;+\;|\mu(z)|\,)\;-\;p\,\lambda\,|\mu(z)|}\;\;\neq 0\,,\quad \text{for all}\;\; z\in\mathbb{D}$$

Furthermore, since $F^{\lambda_\circ}\equiv f\,,\;\; F_z^{\lambda_\circ}\equiv f_z\,,$

$$|\Phi_{\lambda_\circ}(z)| = \;(1+|\mu(z)|\,)\,|f_z| = \,|f_z|\;+\;|f_{\bar z}|\;\;=\;|Df| \tag{6.9}$$

We shall then apply the Interpolation Lemma 3.3 in the measure space $\mathscr{M}(\mathbb{D}\,,\sigma)$ over the unit disk, where

$$\mathrm{d}\sigma(z) = \;\frac{1}{\pi}\Big(1\;-\;\frac{p\,|\mu(z)|}{1+|\mu(z)|}\;\Big)\,\mathrm{d}z$$

We start with the centerpoint $\lambda=0$, where the Beltrami equation reduces to the complex Cauchy-Riemann system $F_{\bar z}\equiv 0$ with principal solution the identity map. Hence $F_z^0\equiv 1\,,\;\Phi_0(z)\equiv 1$ and $M_0=\|\Phi_0\|_\infty=1\,.$

The estimate $M_1\;=\;\sup_{\lambda\in\mathbb{D}}\|\Phi_\lambda\|_2\;\leqslant\;1$ requires just a bit more work. First, in view of Lemma 6.3,

$$\int_{\mathbb{D}} J(z,F^{\lambda})\,\mathrm{d}z\;\;\leqslant\;\;\pi$$

with equality if and only if $F^{\lambda}(z)\equiv z$ outside the unit disk. Here we find from (6.7) that

$$J(z,F^{\lambda}) = |F_z^{\lambda}(z)|^2(1-|\mu_\lambda(z)|^2) = |\Phi_\lambda(z)|^2\left(1-2\,\mathrm{Re}\,\frac{\tau_\lambda(z)}{1+\tau_\lambda(z)}\right) =$$

$$|\Phi_\lambda(z)|^2\left(1-p\frac{|\mu(z)|}{1+|\mu(z)|}\,\mathrm{Re}\,\frac{2\lambda}{1+\lambda}\right)\geqslant|\Phi_\lambda(z)|^2\left(1-p\frac{|\mu(z)|}{1+|\mu(z)|}\right)$$

Hence

$$|\Phi_{\lambda}(z)|^2\;\mathrm{d}\sigma(z)\leqslant\;\frac{1}{\pi}\,J(z,F^{\lambda})\,\mathrm{d}z$$

and, therefore,

$$M_1\;\;=\;\sup_{|\lambda|<1}\;\int_{\mathbb{D}}|\Phi_{\lambda}(z)|^2\;\mathrm{d}\sigma(z)\;\;\;\leqslant\;\sup_{|\lambda|<1}\;\frac{1}{\pi}\int_{\mathbb{D}}J(z,F^{\lambda})\,\mathrm{d}z\;\;\leqslant\;1$$

We are now ready to interpolate. For every $0\leqslant r<1$, in view of the interpolation lemma, we have

$$M_r=\;\sup_{|\lambda|=r}\|\Phi_{\lambda}\|_{\frac{1+r}{r}}\leqslant M_0^{\frac{1-r}{1+r}}\,M_1^{\frac{2r}{1+r}}\;\;\leqslant 1.$$

It remains to substitute $r = \frac{1}{p-1} = \lambda_\circ$. The desired inequality is now immediate,

$$\int_{\mathbb{D}} \left(1 - \frac{p\,|\mu(z)|}{1+|\mu(z)|}\right) \big|Df(z)\big|^p \,\mathrm{d}z = \pi \int_{\mathbb{D}} \big|\Phi_{\lambda_\circ}(z)\big|^{\frac{1+r}{r}} \,\mathrm{d}\sigma(z) \leqslant \pi$$

□

Proof of Theorems 2.1 and 4.1. To infer Theorem 4.1 we extend f as the identity outside Ω. Since Ω is bounded, $\int_\Omega |Df|^2 \leqslant K \int_\Omega J(x,f) < \infty$ so that $f \in \mathscr{W}^{1,2}(\Omega)$. But then one easily verifies that the extended function f defines an element of $\mathscr{W}^{1,2}_{loc}(\mathbb{C})$, and accordingly, a K-quasiconformal map of the entire plane.

Consider then a disk $D_R \supset \Omega$. By re-scaling, if necessary, Inequality (6.3) applies to D_R in place of the unit disk and with $|D_R|$ in place of π. This yields $\mathscr{B}^p_{D_R}[f] \leqslant \mathscr{B}^p_{D_R}[Id]$. On the other hand $\mathscr{B}^p_{D_R\setminus\Omega}[f] = \mathscr{B}^p_{D_R\setminus\Omega}[Id]$, by trivial means. Hence Theorem 4.1 follows.

Theorem 2.1, in turn, is a direct consequence of Theorem 4.1. The pointwise condition $\mathbf{B}_p(Df) \geqslant 0$ for $f = z + h(z)$, together with the boundary condition $h \in \mathscr{C}^\infty_0(\Omega)$, ensures that f represents a (smooth) K-quasiconformal homeomorphism of Ω having identity boundary values. Here p and K are related by $p = 2K/(K-1)$ and we may apply Theorem 4.1 at the borderline exponent. □

7. Sharp $\mathscr{L}\log\mathscr{L}$, $\mathscr{L}^p$ and Exponential integrability

The sharp integral inequalities provided by Theorems 6.4 and 4.1 give us a number of interesting consequences. We start with the following optimal form of the Sobolev regularity of K-quasiconformal mappings.

Corollary 7.1. *Suppose $\Omega \subset \mathbb{C}$ is any bounded domain and $f : \Omega \to \Omega$ is a $K-$quasiconformal mapping, continuous up to $\partial\Omega$, with $f(z) = z$ for $z \in \partial\Omega$. Then*

$$\frac{1}{|\Omega|}\int_\Omega \big|Df(z)\big|^p \,\mathrm{d}z \leqslant \frac{2K}{2K - p\,(K-1)}\,, \quad \textit{for } 2 \leqslant p < \frac{2K}{K-1} \tag{7.1}$$

The estimate holds as an equality for $f(z) = z|z|^{1/K-1}$, $z \in \mathbb{D}$, as well for a family of more complicated maps described in Section 8.2.

Proof. Inequality (7.1) is straightforward consequence of Theorem 4.1, since for $p < 2K/(K-1)$ we have pointwise $\mathbf{B}_p\big(Df(x)\big) \geqslant |Df(x)|^p \, \frac{2K - p\,(K-1)}{2K}$. □

We next introduce yet another rank one-concave variational integral, simply by differentiating $\mathscr{B}^p_\Omega[f]$ at $p = 2$,

$$\mathscr{F}_\Omega[f] := \lim_{p \searrow 2} \frac{\mathscr{B}^p_\Omega[f] \; - \; \mathscr{B}^2_\Omega[f]}{p \; - \; 2} \; = \tag{7.2}$$

$$\frac{1}{2}\int_\Omega \Big[\; \big(1 + \log|Df(z)|^2\big) \; J(z,f) \;\; - \;\; |Df(z)|^2 \Big] \, \mathrm{d}z$$

The nonlinear differential expression $J(z,f)\, \log\, |Df(z)|^2$, for mappings with nonnegative Jacobian, is well known to be locally integrable, see [32] for the following qualitative local estimate on concentric balls $B \subset 2B \subset \Omega$,

$$⨍_B J(z,f)\; \log\left(e + \frac{|Df(z)|^2}{⨍_B |Df|^2}\right) \mathrm{d}z \;\leqslant\; C ⨍_{2B} |Df(z)|^2 \mathrm{d}z \tag{7.3}$$

see also Theorem 8.6.1 in [39]. However, for global estimates one must impose suitable boundary conditions on f. For example, global $\mathscr{L}\log\mathscr{L}(\Omega)$ estimates follow from (7.3) if f extends beyond the boundary of Ω with finite Dirichlet energy and nonnegative Jacobian determinant. This is the case, in particular, when $f(z) - z \in \mathscr{W}^{1,2}_0(\Omega)$.

Let us denote the class of homeomorphisms $f \in \mathscr{W}^{1,2}_{\mathrm{loc}}(\mathbb{C})$ which coincide with the identity map outside a compact set by $\mathscr{W}^{1,2}_{\mathrm{id}}(\mathbb{C})$. It is useful to observe (see e.g. [5, Thm. 20.1.6]) that such a map is automatically uniformly continuous. Further, the $\mathscr{C}^\infty$-smooth diffeomorphisms in $\mathscr{W}^{1,2}_{\mathrm{id}}(\mathbb{C})$ are dense. Precisely, one has

Lemma 7.2 (Approximation Lemma, [38, Thm. 1])**.** *Given any homeomorphism $f \in \mathscr{W}^{1,2}_{\mathrm{id}}(\mathbb{C})$, one can find $\mathscr{C}^\infty$-smooth diffeomorphisms $f^\ell \in \mathscr{W}^{1,2}_{\mathrm{id}}(\mathbb{C})$, $\ell \geqslant 1$, such that*

$$\|f^\ell - f\|_\infty + \; \|D(f^\ell - f\,)\|_{\mathscr{L}^2(\mathbb{C})} \;\to 0, \qquad \text{as } \; \ell \to \infty.$$

Passing to a subsequence if necessary, we may ensure that $Df^\ell \longrightarrow Df$ almost everywhere.

With estimates for the Burkholder integrals we now arrive at sharp global $\mathscr{L}\log\mathscr{L}(\Omega)$ bounds.

Proof of Corollary 2.2. Upon the extension as identity outside Ω, $f \in \mathscr{W}^{1,2}_{\mathrm{id}}(\mathbb{C})$. We use the sequence $\{ f^\ell \}$ in the approximation Lemma 7.2, and view each f^ℓ as a principal solution to its own Beltrami equation

$$f^\ell_{\bar z} \;=\; \mu_\ell(z)\, f^\ell_z \;\;, \qquad |\mu_\ell(z)| \leqslant k_\ell < 1 \;, \quad \mu_\ell(z) = 0 \;, \;\; \text{for} \;\; |z| \geqslant R.$$

where R is chosen, and temporarily fixed, large enough so that $\Omega \subset D_R = \{z \; : |z| < R\}$. It is legitimate to apply Theorem 4.1 for each of the maps $f^\ell : D_R \to D_R$,

$$\mathscr{B}^p_{D_R}[f^\ell] \leqslant \; |D_R| \;=\; \mathscr{B}^2_{D_R}[f^\ell] \;, \qquad \text{whenever} \;\; 2 \leqslant p \leqslant 1 + k_\ell^{-1}$$

Letting $p \searrow 2$ we obtain

$$\int_{D_R} \Big(1 \,+\, \log |Df^\ell(z)|^2 \Big)\; J(z, f^\ell)\, \mathrm{d}z \;\leqslant\; \int_{D_R} |Df^\ell(z)|^2\, \mathrm{d}z \tag{7.4}$$

Convergence theorems in the theory of integrals let us pass to the limit when $\ell \to \infty$, as follows

$$\int_{D_R} J(z,f)\; \big[1 \,+\, \log |Df(z)|^2 \,\big]\; \mathrm{d}z =$$

$$\int_{D_R} J(z,f) \big[1 \,+\, \log(1 + |Df|^2) \big]\; \mathrm{d}z \;-\int_{D_R} J(z,f) \,\big[\log(1 + |Df|^{-2}) \big]\; \mathrm{d}z$$

$$\leqslant \;\; \liminf_{\ell\to\infty} \int_{D_R} J(z,f^\ell)\; \Big[1 \,+\, \log(1 + |Df^\ell|^2) \Big]\; \mathrm{d}z$$

$$- \lim_{\ell\to\infty} \;\; \int_{D_R} J(z,f^\ell)\, \log \Big(1 + |Df^\ell|^{-2} \Big)\; \mathrm{d}z$$

Here the (lim inf)-term is justified by Fatou's theorem while the (lim)-term by the Lebesgue dominated convergence, where we observe that the integrand is dominated point-wise by $J(z,f^\ell)\,|Df^\ell|^{-2} \;\leqslant 1$. The lines of computation continue as follows

$$= \;\; \liminf_{\ell\to\infty} \int_{D_R} J(z,f^\ell)\; \Big[1 \,+\, \log |Df^\ell|^2 \Big]\; \mathrm{d}z \leqslant$$

$$\liminf_{\ell\to\infty} \int_{D_R} |Df^\ell|^2 \; \mathrm{d}z = \int_{D_R} \; |Df(z)|^2\, \mathrm{d}z$$

Finally, we observe that

$$\int_{D_R\setminus\Omega} J(z,f)\; \big[1 \,+\, \log |Df(z)|^2 \,\big]\; \mathrm{d}z = \int_{D_R\setminus\Omega} \; |Df(z)|^2\, \mathrm{d}z \;,$$

which combined with the previous estimate yields (2.3), as desired. $\square$

We next turn to the exponential integrability results, which will follow from Theorem 6.4 at the limit $p \to \infty$.

Proof of Corollary 2.3. Let us assume we are given a function μ, supported in $\mathbb{D}$ with $|\mu(z)| \leqslant 1$ for all $z \in \mathbb{D}$. We then consider the principal solution f of the Beltrami equation $f_{\overline{z}} = \varepsilon \mu f_z$ and apply Theorem 6.4 with $k = \varepsilon$ and $p = 1 + 1/\varepsilon$ to obtain

$$\int_{\mathbb{D}} \left(\frac{1 - |\mu(z)|}{1 + \varepsilon|\mu(z)|} \right) \left| \, Df(z) \, \right|^{1+1/\varepsilon} dz \leqslant \pi. \tag{7.5}$$

By applying the Cauchy-Schwarz inequality and the $\mathscr{L}^2$-isometric property of $\mathbf{S}$, wee see that for almost every $z \in \mathbb{D}$, developing (4.5) to a Neumann series represents f_z as a power series in ε, with convergence radius $\geqslant 1$. Hence

$$f_z = 1 + \varepsilon \mathbf{S}\mu + O(\varepsilon^2) \qquad \text{for a.e. } z \in \mathbb{D}.$$

We may use this to compute pointwise

$$\begin{aligned} (1 + 1/\varepsilon) \log |Df| &= (1 + 1/\varepsilon)\big(\log(1 + \varepsilon|\mu|) + \log |f_z| \big) \\ &= |\mu| + \operatorname{Re} \mathbf{S}\mu + O(\varepsilon). \end{aligned}$$

Hence $\left| \, Df \, \right|^{1+1/\varepsilon} = \exp(|\mu| + \operatorname{Re} \mathbf{S}\mu) + O(\varepsilon)$ and the desired result follows at the limit $\varepsilon \to 0$ by an application of Fatou's lemma on (7.5). □

8. Piecewise Radial Mappings

8.1. **Examples of optimality in Theorems 2.1, 4.1 and 6.4.** Our exposition here is slightly condensed since the basic principle behind these examples can be found already in the paper [11] of Baernstein and Montgomery-Smith, or in the work [37] of Iwaniec . Let us start by describing the building block of the maps that yields equality in our main result. For any $0 \leqslant r < R$ consider the radial map

$$g(z) = \rho(\,|z|\,)\,\frac{z}{|z|} \tag{8.1}$$

defined in the disc $\{|z| \leqslant R\}$. We assume that $\rho : [0, R] \to [0, R]$ is absolutely continuous and strictly increasing with $\rho(0) = 0$, and that ρ is linear on $[0, r]$. We first restrict ourselves to the situation $p \geqslant 1$, and then need the following *expanding assumption*

$$\frac{\rho(t)}{t} \;\geqslant\; \dot{\rho}(t) \;\geqslant 0\,, \qquad t \in (r, R) \tag{8.2}$$

together with the normalization $\rho(R) = R$. Hence on the boundary the map g coincides with the identity map, and if needed we may extend g to the exterior $\{|z| \geqslant R\}$ by setting $g(z) = z$ for these values.

The differential of g exhibits the following rank-one connections

$$Dg(z) = \frac{\rho(|z|)}{|z|} Id + \left(\dot{\rho}(|z|) - \frac{\rho(|z|)}{|z|} \right) \frac{z \otimes z}{|z|^2}. \tag{8.3}$$

It is known, see [15, Proposition 3.4] that concavity along the indicated rank-one lines already secures the quasiconcavity condition for the radial map g. In our situation the assumption (8.2) indeed ensures that the Burkholder integrals become linear on the rank-one segments displayed in (8.3), which implies

$$\mathscr{B}^p_{B(0,R)}[g] = \mathscr{B}^p_{B(0,R)}[Id]. \tag{8.4}$$

Actually, a direct computation (see [11, 37] for details) using the formulas $g_z(z) = \frac{1}{2}\Big(\dot{\rho}(|z|) + \rho(|z|)/|z| \Big)$ and $g_{\bar{z}}(z) = \frac{1}{2}\Big(\dot{\rho}(|z|) - \rho(|z|)/|z| \Big) z/\bar{z}$ yields

$$\begin{aligned} \mathscr{B}^p_{B(0,R)}[g] &= \pi \int_0^R \left(\frac{[\rho(t)]^p}{t^{p-2}} \right)' \mathrm{d}t = \pi \frac{[\rho(R)]^p}{R^{p-2}} - \lim_{t \to 0^+} \pi \frac{[\rho(t)]^p}{t^{p-2}} \\ &= \pi R^2 = |B(0,R)| = \mathscr{B}^p_{B(0,R)}[Id]. \end{aligned}$$

The above computation indicates that if $r = 0$, we must in addition require

$$\rho(t) = o(t^{1-\frac{2}{p}}) \qquad \text{as} \quad t \to 0\,. \tag{8.5}$$

Assume then that $f_0(z) = az + b$ is a (complex) linear map defined in a bounded domain $\Omega \subset \mathbb{C}$. Given $0 \leqslant r < R$ and a ball $B(z_0, R) \subset \Omega$ together with the increasing homeomorphism $\rho : [0, R] \to [0, R]$ and the radial map g as discussed above, we may modify f_0 in $B(z_0, R)$ by defining

$$f_1(z) = \begin{cases} f_0(z) & \text{if } \ z \notin \overline{B(z_0, R)}, \\ a g(z - z_0) + (a z_0 + b) & \text{if } \ z \in B(z_0, R), \end{cases}$$

By scaling, (8.4) shows that we have $\mathscr{B}^p_{B(z_0,R)}[f_1] = \mathscr{B}^p_{B(z_0,R)}[f_0]$, and, consequently

$$\mathscr{B}^p_{\Omega}[f_1] = \mathscr{B}^p_{\Omega}[f_0]. \tag{8.6}$$

In the next step we may deform f_1 in a disc that is contained in either one of the sets $\Omega \setminus B(z_0, R)$ or $B(z_0, r)$, where f_1 is linear. Inductively one obtains f_n from f_{n-1} by deforming f_{n-1} accordingly in the domains of linearity. By induction, we see that all such mappings have the same energy, which is equal to the energy of their linear boundary data $az+b$:

$$\mathscr{B}^p_\Omega\,[f_n] \;=\; \mathscr{B}^p_\Omega\,[az+b] \;=\; a^p\,|\Omega| \tag{8.7}$$

This iteration process may, but need not, continue indefinitely so as to arrive at e.g. Cantor type configuration of annuli and a homeomorphism $f_\infty : \Omega \xrightarrow{\text{onto}} \Omega^* := a\Omega + b$. Without going to the formal definition, we loosely refer to reasonable (e.g. converging in $\mathscr{W}^{1,1}$) such limits $f = \lim_{n\to\infty} f_n$ as *piece-wise radial mappings*, see Figure 2.

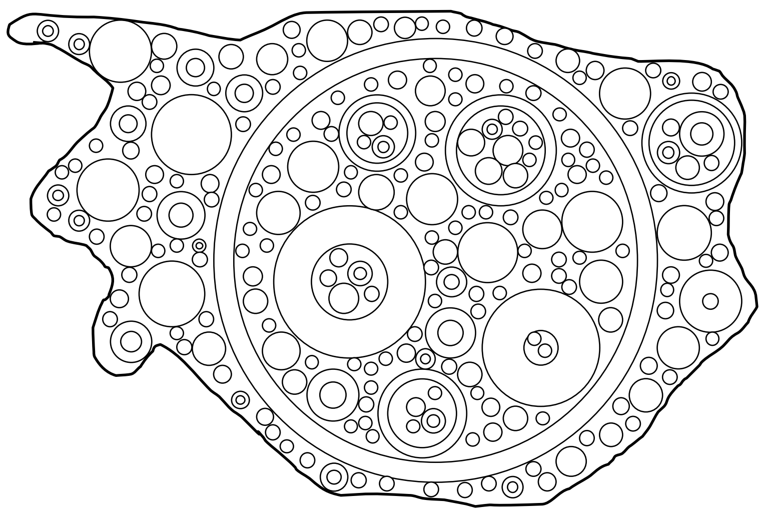

Figure 2. Annular Packing.

Definition 8.1. Let $p \geqslant 1$ and let $\Omega \subset \mathbb{C}$ be any nonempty bounded domain. The class $\mathscr{A}^p(\Omega)$ consists of piece-wise radial mappings f_∞ : $\Omega \xrightarrow{\text{onto}} \Omega$ whose construction starts with $f_0(z) \equiv z$, the convergence $f_\infty = \lim_{n\to\infty} f_n$ takes place in $\mathscr{W}^{1,p}(\Omega)$ and the condition (8.2) is in force.

The following observation is a direct corollary of (8.7).

Proposition 8.2. *For any $p \geqslant 1$ and $f \in \mathscr{A}^p(\Omega)$ we have*

$$\mathscr{B}^p_\Omega\,[f] = \mathscr{B}^p_\Omega\,[\mathrm{Id}] \;= |\Omega|. \tag{8.8}$$

In interpreting this conclusion one may say that $\mathscr{B}^p_\Omega$ is a null Lagrangian when restricted to $\mathscr{A}^p(\Omega)$.

We get a plethora of fairly complicated maps that produce equality in our main Theorems. One just needs to consider any $f \in \mathscr{A}^p$ that satisfies the additional condition $\mathbf{B}_p(Df(x)) \geqslant 0$, i.e. in the construction one applies maps ρ that satisfy

$$\frac{\rho(t)}{t} \geqslant \dot{\rho}(t) \geqslant \left(1 - \frac{2}{p}\right) \frac{\rho(t)}{t}. \tag{8.9}$$

In case of Theorem 2.1, in order to satisfy the smoothness assumption one of course has to pick the functions ρ in the construction so that the (possibly limiting) map belongs to $Id + \mathscr{C}_\circ^\infty(\Omega)$.

In view of Conjecture 1.1, the maps in $\mathscr{A}^p(\Omega)$ are potential global extremals for $\mathscr{B}_\Omega^p$. Indeed, it can be shown that they are critical points of the associated Euler-Lagrange equations. Furthermore, this property to a large extent characterizes Burkholder functionals: these functionals are the only (up to scalar multiple) isotropic and homogeneous variational integrals with the $\mathscr{A}^p(\Omega)$ as their critical points [9].

Here, we content with pointing out the following result, where we employ the customary notation $\mathscr{C}^1_{\mathrm{id}}(\Omega) = \mathrm{id} + \mathscr{C}^1_\circ(\Omega)$, where $\mathscr{C}^1_\circ(\Omega)$ stands for the space of functions $\mathscr{C}^1$ -smooth up to the boundary of Ω and vanishing on $\partial\Omega$.

Corollary 8.3. *The Burkholder functional* $\mathscr{B}_\Omega^p : \mathscr{C}^1_{\mathrm{id}}(\Omega) \to \mathbb{R}$, $p > 2$, *attains its local maximum at every* $\mathscr{C}^1$*- smooth piece-wise radial map in* $\mathscr{A}^p(\Omega)$ *for which the condition* (8.9) *is further reinforced to:*

$$\frac{\rho(t)}{t} \geqslant \dot{\rho}(t) \geqslant \frac{1}{K} \frac{\rho(t)}{t}, \qquad K < \frac{p}{p-2}. \tag{8.10}$$

Proof. The lower bound (8.10) can be used to verify that f is K-quasiconformal with $K < p/(p-2)$. Namely, as $f \in \mathscr{C}^1_{\mathrm{id}}(\Omega)$, one checks that f is necessarily conformal at points corresponding to $t = 0$ and the derivative is non-vanishing. Hence the strict inequality for K will not be destroyed by small $\mathscr{C}^1$ -perturbations. Theorem 4.1 applies, and by combining it with Proposition 8.2 the claim is evident. □

8.2. **Equality in Corollaries 2.2, 2.3 and 7.1 .** If one substitutes in the formula (7.2) a function for which $\mathscr{B}_\Omega^p[f] = \mathscr{B}_\Omega^2[f]$, we acquire the equality in the $\mathscr{L}\log\mathscr{L}$-inequality of Corollary 2.2. Especially, by (8.8) we obtain

Lemma 8.4. *Let Ω be a bounded domain in the plane. If f belongs to class $\bigcup_{p>2} \mathscr{A}^p(\Omega)$, then there is equality in* (2.3) *in Corollary 2.2.*

Actually, one checks that in the construction condition (8.5) can be replaced by the analogue $\rho(t) = o\big(\log(1/t)^{-1}\big)$.

We next turn our attention to Corollary 2.3. It turns out that there, as well, one has a very extensive class of functions μ of radial type that yield an equality in the estimates. These functions can be viewed as infinitesimal generators of the expanding class of radial mappings defined above.

Lemma 8.5. *Let $\alpha\colon (0,1) \to [0,1]$ be measurable and with the property*

$$\int_0^1 \frac{1-\alpha(t)}{t} dt = \infty.$$

Set

$$\mu(z) = -\frac{z}{\bar{z}}\alpha(|z|) \quad \textit{for } |z| < 1, \qquad \mu(z) = 0 \quad \textit{for } |z| \geqslant 1.$$

Then there is equality in Corollary 2.3, i.e.

$$\int_{\mathbb{D}} (1 - |\mu(z)|)\, e^{\,|\mu(z)|}\, \big|\exp(\mathbf{S}\mu(z))\big|\, dz = \pi$$

Proof. Let $\phi(z) = 2z \int_{|z|}^1 \frac{\alpha(t)}{t}\, dt$ for $|z| < 1$ and set $\phi(z) = 0$ elsewhere. Then we compute that $\phi \in \mathscr{W}^{1,2}(\mathbb{C})$ with

$$\phi_{\bar{z}} \equiv \mu, \qquad \phi_z = \mathbf{S}\mu(z) = 2\int_{|z|}^1 \frac{\alpha(t)}{t}\, dt - \alpha(|z|), \quad |z| < 1.$$

Thus

$$\int_{\mathbb{D}} (1 - |\mu(z)|) e^{|\mu(z)| + \mathrm{Re}\, \mathbf{S}\mu(z)} dm$$
$$= 2\pi \int_0^1 \big(1 - \alpha(t)\big) \exp\left[2\int_t^1 \frac{\alpha(s)}{s}\, ds\right]\, t\, dt = \pi,$$

as we have the identity

$$\frac{d}{dt}\left(t^2 \exp\left[2\int_t^1 \frac{\alpha(s)}{s}\, ds\right]\right) = 2t\big(1 - \alpha(t)\big) \exp\left[2\int_t^1 \frac{\alpha(s)}{s}\, ds\right]$$

and our assumption gets rid of the substitution at $t = 0$. □

More complicated examples may be obtained by a similar iteration procedure as described above.

Finally, equality in (7.1) obviously implies that necessarily the distortion function $K(z,f) \equiv K$ in Ω. Hence examples are produced by specific functions ρ in (8.1), the powers

$$\rho_K(t) \; = \; R^{1-1/K}\, t^{1/K}, \quad r < t < R,$$

where $\rho_K(t)$ is linear on $(0,r]$, if $r > 0$. For $2 \leqslant p < \frac{2K}{K-1}$ let $\mathscr{A}^p_K(\Omega)$ denote the subclass of $\mathscr{A}^p(\Omega)$ consisting of those piecewise radial mappings where, first, we fill the domain Ω by discs or annuli up to measure zero, second, at each construction step choose $\rho = \rho_K$, and third, choose $r = 0$ at any possible subdisk remaining in the limiting packing construction. This ensures that the limiting function f does not remain linear in any subdisk, so that we have $K(z,f) \equiv K$ up to a set of measure zero. Then, as $|\Omega| < \infty$, it is easy to see that convergence $f_\infty = \lim_{j\to\infty} f_j$ takes place in $\mathscr{W}^{1,p}$ since now $1 - p|\mu_j(z)|(1+|\mu_j(z)|)^{-1} \geqslant c_0 > 0$. Moreover, since there is equality in Theorem 6.4 and $K(z,f) \equiv K$, one obtains for any $f \in \mathscr{A}^p_K(\Omega)$ that

$$\frac{1}{|\Omega|}\int_\Omega \big|Df(z)\big|^p \, \mathrm{d}z = \; \frac{2K}{2K \; - \; p\,(K-1)} \; .$$

9. RANK-ONE CONCAVE ENVELOPES

Definition 9.1. Given a continuous function $\mathbf{E} : \mathbb{R}^{m\times n} \to \mathbb{R}$, we use a visual notation to define:

- *Rank-one concave envelope* of $\mathbf{E}$ (*the smallest majorant*) as,

$$\mathbf{E}^{\frown}_R = \; \inf\{\, \Xi\,; \;\; \Xi : \; R^{m\times n} \to \mathbb{R} \; \text{ is rank-one concave, and } \Xi \geqslant \mathbf{E}\}$$

- *Quasiconcave envelope* of $\mathbf{E}$ as,

$$\mathbf{E}^{\frown}_Q \; = \; \inf\{\, \Xi\,; \;\; \Xi : \; R^{m\times n} \to \mathbb{R} \; \text{ is quasiconcave, and } \Xi \geqslant \mathbf{E}\}$$

Obviously $\mathbf{E}^{\frown}_Q \geqslant \mathbf{E}^{\frown}_R$ pointwise; the former function being quasiconcave and the latter rank-one concave.

Theorem 9.2. *Recall the Beurling function* $\mathbf{F}_p : \mathbb{C}\times\mathbb{C} \to \mathbb{R}$

$$\mathbf{F}_p\,(\xi,\zeta) \;\overset{\text{def}}{=\!=}\; |\,\xi|^p - (p^* - 1)^p\,|\,\zeta|^p \;, \qquad 1 < p < \infty\,,$$

and the Burkholder's function

$$\mathbf{B}_p\,(\xi,\zeta) \;\overset{\text{def}}{=\!=}\; \big[\,|\,\xi| - (p^* - 1)\,|\,\zeta|\,\big]\cdot\big[\,|\,\xi| \,+\, |\,\zeta|\,\big]^{p-1}\,.$$

Then the rank-one concave envelope of $\mathbf{F}_p$ *is given by the following formula. For* $p \geqslant 2$*,*

$$\mathbf{F}_p^{\frown}(\xi,\zeta) \;=\; \begin{cases} |\,\xi|^p - (p^*-1)^p|\zeta|^p \;=\; \mathbf{F}_p(\xi,\zeta) & \textit{if} \quad (p^*-1)\,|\zeta| \;\geqslant\; |\xi| \\ p\,(1-1/p^*)^{p-1}\;\mathbf{B}_p & \textit{if} \quad (p^*-1)\,|\zeta| \;\leqslant\; |\xi| \end{cases}$$

While, for $1 < p < 2$*,*

$$\mathbf{F}_p^{\frown}(\xi,\zeta) \;=\; \begin{cases} p\,(1-1/p^*)^{p-1}\;\mathbf{B}_p & \textit{if} \quad (p^*-1)\,|\zeta| \;\geqslant\; |\xi| \\ \mathbf{F}_p(\xi,\zeta) & \textit{if} \quad (p^*-1)\,|\zeta| \;\leqslant\; |\xi| \end{cases}$$

Burkholder [26] shows this in a slightly different sense. Namely, that the envelope function above is the smallest majorant of $\mathbf{F}_p$ which is concave in orientation-reversing directions (as discussed on page 5). See also, p. 64 in [17]. The result as stated here basically follows from the work [54].

Proof. Let us denote by $\mathbf{E}(\xi,\zeta)$ the formula given above. Our task is to show that $\mathbf{F}_p^{\frown} = \mathbf{E}$. For any pair $\theta_1, \theta_2 \in [0,\pi)$, consider the function $\mathbf{F}_{p,\theta_1,\theta_2}^{\frown} \colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$,

$$(x,y) \mapsto \mathbf{F}_p^{\frown}(e^{i\theta_1}x, e^{i\theta_2}y).$$

Using rank-one concavity of $\mathbf{F}_p^{\frown}$ we see that $\mathbf{F}_{p,\theta_1,\theta_2}^{\frown}$ is zig-zag concave, that is, concave in the directions of $\pm\,\pi/4$ in $\mathbb{R}^2$. By the results (Theorem 6 and 7) of [54] on the zig-zag concave envelope of $|x|^p - (p^*-1)^p|y|^p$, we have that $\mathbf{F}_{p,\theta_1,\theta_2}^{\frown}(x,y) \geqslant \mathbf{E}(|x|,|y|)$. Since, this is true for any $\theta_1, \theta_2 \in [0,\pi)$ we have the inequality $\mathbf{F}_p^{\frown}(\xi,\zeta) \geqslant \mathbf{E}(|\xi|,|\zeta|) = \mathbf{E}(\xi,\zeta)$. On the other hand, as we have remarked $\mathbf{E}$ is rank-one concave so $\mathbf{F}_p^{\frown} = \mathbf{E}$ as claimed. □

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