

NAFSA 10

Trest, June 9-15, 2014

Giovanni Alberti

RECTIFIABLE MEASURES AND APPLICATIONS

1. Introduction

In these two lectures I will explain a few results obtained in joint works with Marianna Csörnyei (Chicago), Andrea Marchese (MPI, Leipzig) and David Preiss (Warwick).

These results stemmed from a research started few years ago together with M. Csörnyei and D. Preiss, when we noticed that the solutions of a bunch of (apparently unrelated) problems had the same common root.

Thus my plan is to start straight away with a list of some of these problems, and then set up the framework for the solution.

Problem 1 Closability of the gradient operator (w.r.t. a singular measure).

As it is well-known, there are many (essentially equivalent) ways to define Sobolev spaces $W^{1,p}$ on an open domain

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in \mathbb{R}^n . One is to use the integration-by-parts formula (or the divergence theorem), which leads to the notion of gradient in the sense of distribution. Another approach is to define $W^{1,p}$ as the completion of the space of C^1 (or smooth) functions with respect to the Sobolev norm $\|u\|_{W^{1,p}} := \|u\|_p + \|\nabla u\|_p$.

Assume now that you want to define $W^{1,p}$ on \mathbb{R}^n endowed with a measure μ which is not the Lebesgue measure (or, more generally, a metric space endowed with some measure). In this case the first approach will not be viable, due to a lack of a suitable integration - by - parts formula.

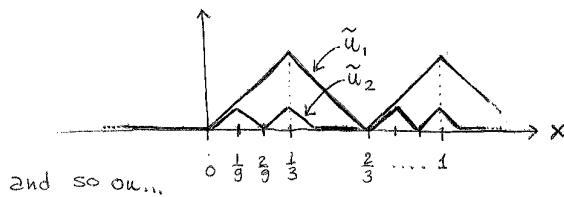
The second approach, on the other hand, will always work, but there is an hidden subtlety: Consider a Cauchy sequence (u_n) of smooth functions (where "Cauchy" refers to the Sobolev norm defined as above, with μ in place of the Lebesgue measure); then both (u_n) and (∇u_n) are Cauchy sequences in L^p , and therefore converge to some limits, say u and v .

Now one would like to say that v is the gradient of u . But here is a problem: in principle different sequences generating the same u may give different v , which means that the gradient of u is not actually well-defined (or, if you prefer, that the Banach space $W^{1,p}$ constructed this way is a rather abstract space).

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Example

Consider indeed the following example: μ is a measure supported on the Cantor set $C \subset \mathbb{R}$, then the constant function $u=0$ can be obtained as limit of the (trivial) sequence $u_n=0 \ \forall n$, but also of \tilde{u}_n given as follows:



That is, writing $C = \bigcap C_n$ where each C_n is the union of 2^n intervals of length $1/3^n$, as usual, then \tilde{u}_n is defined by

$$\tilde{u}'_n := \begin{cases} 1 & \text{on } C_n, \\ -1 & \text{on } C_n + 1/3^n, \\ 0 & \text{elsewhere.} \end{cases}$$

One easily checks that \tilde{u}_n is a Cauchy sequence (in the Sobolev norm...) but $\tilde{u}'_n \rightarrow 1$ in $L^p(\mu)$ for every p . (while $u'_n \rightarrow 0$, obviously).

Note that the fact that the functions \tilde{u}_n are Lipschitz and not smooth can be easily fixed, and that the same construction works with any measure μ supported on any totally disconnected compact set C .

In particular μ could be taken absolutely continuous w.r.t. the Lebesgue measure!

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In the language of Functional Analysis, the phenomenon illustrated by the previous example can be restated by saying that the gradient (or derivative) operator is not closable. (More precisely, if G is the graph of the gradient operator from \mathcal{E}^1 to \mathcal{E}^0 , viewed as a subset of $L^p(\mu) \times L^p(\mu)$, then its closure is no longer the graph of a univalued operator).

So, the problem arises of characterizing the measures μ for which the gradient operator is closable.

To this regard, it was conjectured that if the gradient is closable then μ must be absolutely continuous w.r.t. Lebesgue measure (even though this condition is by no means sufficient, as the example above shows).

This conjecture is easy to prove in dimension $m=1$.

Together with M. Csörnyei and D. Preiss we proved this conjecture for $m=2$ (and this will be explained in detail in these lectures).

Moreover, our approach together with another result announced recently by M. Csörnyei and Peter Jones proves the conjecture in any dimension m .

In other words we show that if μ is a measure on \mathbb{R}^m , not absolutely continuous w.r.t. the Lebesgue measure, then there exists a sequence of smooth functions (u_n) such that $u_n \rightarrow 0$ uniformly and $\nabla u_n \rightarrow g \neq 0$ in $L^p(\mu)$.

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Remark

The conjecture about closability stated above is strictly related to questions with a similar flavour that were considered at a certain moment within other communities. One is the following: let (f_n) be a sequence of smooth maps from (∞ bounded domain of) \mathbb{R}^m to \mathbb{R}^n which are uniformly Lipschitz and converge uniformly to some smooth f . Then it is known that the Jacobian determinants $\det(\nabla f_n)$ converge weakly to $\det(\nabla f)$ (more precisely, weakly* in L^∞).

The same holds if we replace the Lebesgue measure in L^∞ with a measure μ which is absolutely continuous w.r.t. the Lebesgue measure. But what happens if μ is not a.c.? Are there μ , not a.c., such that the property above still holds?

The answer, once again, turns out to be NO. (Again, for $m=2$ this follows directly from the work of myself, M.C. and D.P., while for $m>2$ it requires the result by M.C. and P.J. mentioned above).

Problem 2 Rank-one property of derivatives of BV maps

Given a map $u: \mathbb{R}^m \rightarrow \mathbb{R}^n$ in L^1 , we say that u is of class BV (bounded variation) if its distributional derivative (gradient) is represented by a measure on \mathbb{R}^m with values in $m \times n$ matrices, or

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equivalently,

$$Du = g \cdot \mu$$

where μ is a positive measure on \mathbb{R}^m and g is a density with values in $m \times n$ matrices such that $g \neq 0$ μ -a.e. (and $g \in L^1(\mu)$).

Then, denoting by μ_s the singular part of μ wrt. \mathcal{L}^m , we have that

$$\text{rank}(g(x)) = 1 \text{ for } \mu_s\text{-a.e. } x.$$

This fact has a certain technical relevance; it was conjectured by E. De Giorgi and L. Ambrosio, and then proved by myself back in '93.

Another (hopefully simpler) proof will be given below.

Problem 3 Extension of Rademacher theorem to measures different from the Lebesgue measure

As we all know, Rademacher theorem states that a Lipschitz function on \mathbb{R}^m is differentiable a.e., and "a.e." refers to the Lebesgue measure.

But what happens if we replace the Lebesgue measure with another measure μ ?

Of course, if μ is a.c. wrt. \mathcal{L}^m then Rademacher Theorem still holds, but what if μ is singular?

It is easy to give examples of singular measures μ for which Rademacher theorem fails, but are there some for which it holds?

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The answer is always negative; this is easy to prove in dimension $m=1$, it is due to [ACP] for $m=2$, and is extended to every dimension thanks to [CJ].

Actually, in the following consider a refined version of this problem, namely: given a (possibly singular) measure μ , how should we modify the statement of Rademacher theorem to make it true?

Let me give an example: if E is curve of class C^1 in \mathbb{R}^m (intended as a submanifold, not a parametrization) and μ is the length measure on E , then it is easy to show (using a parametrization and Rademacher theorem in one variable) that

- every Lipschitz function $f: \mathbb{R}^m \rightarrow \mathbb{R}$ is differentiable μ -a.e. in the direction tangential to E ;
- the previous result is optimal, in the sense that there exists a Lipschitz f such that at μ -a.e. x f is not differentiable in any direction v which is not tangential to E (at x), take for example $f(x) := \text{dist}(x, E)$.

We will see that a similar statement holds for an arbitrary measure μ , given the "right" notion of tangential bundle (for a measure).

This is actually the core of these lectures.

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Before starting with definitions, a few comments.

The theory I am going to explain can be developed by keeping the focus on (singular) measures or on (Lebesgue null) sets. The latter approach gives results that are, in some sense, more precise, and this is exactly the approach followed by [ACP].

On the other hand the former approach is in a way simpler, and is the one I will follow in this lecture, and it was pursued by myself and my (then) PhD student Andrea Marchese based on the previous work by [ACP].

Let me quickly explain what may be the difference between "sets" and "measures", with an example:

let A_1, A_2 be Borel sets in \mathbb{R}^m , and let f_1, f_2 be Lipschitz functions on \mathbb{R}^m s.t. f_i is not differentiable at any point of A_i .

Can we then construct a Lipschitz function f which is not differentiable on $A_1 \cup A_2$?

If you think about it, you will soon realize that the answer is not immediately yes, and actually turns out that (at least for me, surprisingly) the answer is no.

On the other hand, if you assume that f_i is not differentiable at μ -a.e. point of A_i , and look for an f which is not differentiable at μ -a.e. point of $A_1 \cup A_2$, then it is an easy exercise to show that a random linear combination $f = \alpha_1 f_1 + \alpha_2 f_2$ will do! (random wrt. the Lebesgue measure on the set of all

$(\alpha_1, \alpha_2) \in [0, 1]^2$, for example).

⑨

References

[ACP] = G. Alberti - M. Csörnyei - D. Preiss.

Many results have been described in two survey papers:

Structure of null sets in the plane and applications, in

"Proceedings of the 4th ECM,, European Math. Soc.,
Zürich, 2005;

Differentiability of Lipschitz functions, structure of null
sets, and other problems, in "Proceedings of the
ICM 2010,, Hindustan Book Agency, New Delhi, 2010.

The main paper has been postponed for many years,
due to the (suspected and now confirmed) suboptimality
of certain definitions. This point has been clarified
only recently, thanks to Andras Mathe.

[AM] = G. Alberti - Andrea Marchese.

On the differentiability of Lipschitz functions with
respect to measures on the Euclidean space.

This paper should be completed soon.

[CJ] = M. Csörnyei - Peter Jones.

Their result has only been announced.

2. Decomposability bundle

As pointed out before, our purpose is to identify, for a given measure μ on \mathbb{R}^m (finite, positive, Borel...) and for μ -a.e. x , the largest possible set of directions $V(\mu, x)$ such that every Lipschitz function f on \mathbb{R}^m is differentiable in every direction in $V(\mu, x)$ for μ -a.e. x .

Let us begin by a simple remark.

Assume that μ is of the form

$$(1) \quad \mu = \int_I \mu_t dt$$

where $I := [0, 1]$ and dt stands for the Lebesgue measure on I , each μ_t is the length measure on some curve E_t , and the integral above means that $\mu(E) = \int_I \mu_t(E) dt$ for every Borel set E .

Assume moreover that there exists a (Borel, unitary) vector field τ s.t.

$\tau(x)$ is tangent to E_t at x
for a.e.t and μ -a.e.x.

Then it is immediate that every Lipschitz function f is differentiable at x in the direction $\tau(x)$ for μ -a.e.x.

Indeed f is differentiable in the direction $\tau_t(x)$ tangent to E_t at x for μ_t -a.e.x and every t (as already pointed out before), moreover $\tau(x) = \pm \tau_t(x)$ for μ_t -a.e.x and a.e.t, and finally we use that " μ_t -a.e.x and a.e.t" means " μ -a.e.x".

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The previous observation suggests that the bundle $V(\mu, x)$ we are looking for should be related to all possible decompositions of μ (or "parts of μ ") as above. More precisely we would like to define $V(\mu, x)$ as the smallest vector space such that

$V(\mu, x)$ contains $Tan(E_t, x)$
for μ_t -a.e.x and a.e.t

for every decomposition of μ as above.

Now, as written this definition does not make sense, because it's pointwise in nature, while there are too many "a.e." around. Moreover it is technically convenient to consider a larger class of decomposition.

We need therefore to introduce some notation.

Rectifiable and purely unrectifiable sets

We say that a (Borel !) set $E \subset \mathbb{R}^m$ is rectifiable if it has finite length ($H^1(E) < +\infty$) and $E = \bigcup_{n=0}^{\infty} E_n$ where

- E_0 is H^1 -null (here H^1 is the 1-dimensional Hausdorff measure \approx the "length" measure);
 - each E_n with $n > 0$ is contained in the image of a Lipschitz path $\gamma_n : I$ (interval) $\rightarrow \mathbb{R}^m$.
- It can be shown that this definition remains the same if we replace the second condition by
- each E_n is contained in a curve C_n of class C^1 (intended as a submanifold of \mathbb{R}^m).

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Moreover a rectifiable set E admits a tangent bundle $\text{Tan}(E, x)$ in the following weak sense: $\text{Tan}(E, x)$ is a line for every $x \in E$, and for every curve C of class C^1 there holds

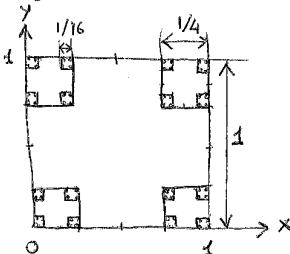
$$\text{Tan}(E, x) = \text{Tan}(C, x) \text{ for } H^1\text{-a.e. } x \in E \cap C.$$

This properties identify the bundle $\text{Tan}(E, x)$ uniquely (up to an H^1 -null subset of E), and of course this weak tangent bundle agrees with the usual one if E is a curve of class C^1 ...

Finally we say that a set E is purely unrectifiable if $H^1(E \cap C) = 0$ for every curve C of class C^1 .

I tend to picture rectifiable sets as finite unions of curves, but be aware that they can be rather wild, for example $E = \bigcup I_n$ where each I_n is a segment with length at most 2^{-n} chosen in such a way that the midpoints x_n and the directions τ_n are dense (in $\mathbb{R}^m \times \mathbb{R}^m$).

An example of purely unrectifiable set (with positive length) is the following Cantor type set in the plane



$$E = \bigcap E_n$$

where each E_n is the union of 4^n squares with side-length $1/4^n$, as in the picture.

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Indeed it can be shown that $H^1(E) > 0$, and is purely unrectifiable, that is $H^1(E \cap C) = 0$ for every curve C of class C^1 . Indeed we can restrict to the case C is the graph of a C^1 function $y = y(x)$ (or $x = x(y)$), and

$$H^1(E \cap C) = \int |\dot{y}| dx$$

π_i := projection
on the x axis

$$\leq \int |\dot{y}| dx = 0$$

because $\pi_i(E)$ has measure 0.

Actually this argument shows that every set $E \subset \mathbb{R}^2$ with null projections on the axes (any two axes, actually) is purely unrectifiable, and this class of sets includes sets with dimension up to 2 (unclucted).

For more details about rectifiable and purely unrectifiable sets see, e.g., [Krantz & Parks: Geometric Integration Theory] or [Morgan: Geometric Measure Theory, a beginner's Guide].

Integration of measures

By measure we usually mean a positive measure on the appropriate Borel σ -algebra, and we usually assume that it is finite (\mathcal{L}^n and H^1 being two of the few exceptions). Given a family (μ_t) of such measure (say, on \mathbb{R}^m) with $t \in I := [0, 1]$, we would

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like to define $\mu = \int_I \mu_t dt$ by

$$\mu(E) := \int_I \mu_t(E) dt \text{ for every Borel set } E \in \mathbb{R}^m.$$

For the definition to make sense (and for μ to be finite) (μ_t) must actually satisfy some additional requirements:

- $t \mapsto \mu_t(E)$ is Borel for every E
- $t \mapsto \|\mu_t\| := \mu_t(\mathbb{R}^m)$ has finite integral.

But we won't verify these properties in the following...

In principle we can consider a more general parameter space I and a more general measure dt (and we will sometimes do it in the following) but recall that by a suitable reparametrization we can always reduce to the case $I = [0,1]$ and $dt = \text{Lebesgue meas. on } I$ (at least for any reasonable choice of I and dt).

Definition of decomposability bundle

Let μ be a measure on \mathbb{R}^m . Let then \mathcal{G}_μ be the class of all families (μ_t) as above such that

- each μ_t is the restriction of \mathcal{H}^1 to some rectifiable set E_t ;
- $\int_I \mu_t dt$ is absolutely continuous w.r.t. μ .

Let then \mathcal{G}_μ be the class of all Borel maps $x \mapsto V(x)$ where $V(x)$ is a subspace of \mathbb{R}^m such that

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- $T_{\mathcal{H}^1}(E_t, x) \subset V(x)$ for μ_t -a.e. x (that is, \mathcal{H}^1 -a.e. $x \in E_t$) and a.e.t.

Then \mathcal{G}_μ admits an element $x \mapsto \bar{V}(x)$ which is minimal w.r.t. inclusion μ -a.e. (that is, for every $x \mapsto V(x) \in \mathcal{G}_\mu$ there holds $\bar{V}(x) \subset V(x)$ for μ -a.e. x). This minimal element is unique (up to a μ -null set) and we call it decomposability bundle of μ , denoted $x \mapsto V(\mu, x)$.

Remark When we say that $x \mapsto V(x)$ is Borel, we intend it as a map from \mathbb{R}^m to the space $\mathcal{G}(m)$ given by the (topologically disjoint) union of the Grassmannians $\mathcal{G}(m, k)$ with $k=0, \dots, m$.

The existence and uniqueness of the minimal element of \mathcal{G}_μ is a consequence of the following lemma: let \mathcal{G} be a family of maps $x \mapsto V(x) \in \mathcal{G}(m)$ (Borel maps, of course) which is closed under countable intersection (that is, if $x \mapsto V_n(x)$ belongs to \mathcal{G} for every $n=1, 2, \dots$ then also $x \mapsto \bigcap_n V_n(x)$ belongs to \mathcal{G}); then \mathcal{G} admits a minimal element, which is unique (in the sense specified above).

We can now state our main result concerning the differentiability of Lipschitz functions w.r.t. μ (see [AM]):

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Theorem 1 (Rademacher theorem for measures)

Let μ and $x \mapsto V(\mu, x)$ be as above. Then

- (i) For every Lipschitz function $f: \mathbb{R}^m \rightarrow \mathbb{R}$ and μ -a.e. x , f is differentiable at x w.r.t. the plane $V(\mu, x)$, that is, the following 1st order Taylor expansion holds:

$$f(x+h) = f(x) + d_V f(x) \cdot h + O(|h|)$$

where $d_V f(x)$ (the differential of f at x w.r.t. $V(\mu, x)$) is a linear function from $V(\mu, x)$ to \mathbb{R} .

- (ii) Moreover the previous statement is sharp, in the sense that there exists $f: \mathbb{R}^m \rightarrow \mathbb{R}$ Lipschitz such that, for μ -a.e. x and every $v \notin V(\mu, x)$ the partial derivative $\frac{\partial f}{\partial v}(x)$ does not exist.

This theorem will be proved later.

Now I will instead make some comments on the previous statement (and definition).

Remarks

(yes, I know, it's not finite...)

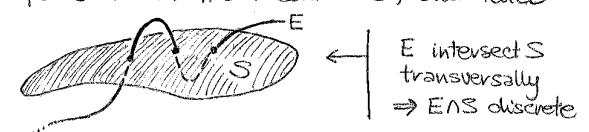
- (i) If $\mu = \mathcal{L}^m$, then $V(\mu, x) = \mathbb{R}^m$ for μ -a.e. x . Consider for simplicity the case $m=2$. One decomposition of the Lebesgue measure μ is $\mu = \int_{\mathbb{R}^2} \mu_t dt$ where μ_t is the length measure of the "vertical" line R_t passing through $(t, 0)$. (This is just Fubini's Theorem.) Thus $(0, 1) \in V(\mu, x)$ for μ -a.e. x .

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On the other hand, by a similar argument we have that $(1, 0) \in V(\mu, x)$ for μ -a.e. x as well, and the proof is concluded.

- (ii) If μ is absolutely cont. w.r.t. \mathcal{L}^m ($\mu \ll \mathcal{L}^m$) then $V(\mu, x) = \mathbb{R}^m$ for μ -a.e. x as well. This follows from Remark (i) and the following "locality principle", (which is not hard to prove): if $\tilde{\mu} \ll \mu$ then $V(\tilde{\mu}, x) = V(\mu, x)$ for $\tilde{\mu}$ -a.e. x .

- (iii) If μ is the restriction of the k -dimensional Hausdorff measure \mathcal{H}^k to a k -dimensional surface S of class \mathcal{C}^1 , then $V(\mu, x) = \text{Tan}(S, x)$ for μ -a.e. x (that is, \mathcal{H}^k -a.e. $x \in S$). The argument to prove the inclusion $V(\mu, x) \supset \text{Tan}(S, x)$ is a variant of the one in Remark (i). To prove the opposite inclusion we need to show that given a family (E_t) in \mathcal{F}_μ , then $\text{Tan}(E_t, x) \subset \text{Tan}(S, x)$ for μ -a.e. x (that is, \mathcal{H}^1 -a.e. $x \in E_t$). This is a consequence of the following fact: if E is a rectifiable set and S a k -surface, then $\text{Tan}(E, x) \subset \text{Tan}(S, x)$ for \mathcal{H}^1 -a.e. $x \in E \cap S$. And actually it suffices to prove this statement when E is a curve of class \mathcal{C}^1 , and this follows by the fact that when $\text{Tan}(E, x) \not\subset \text{Tan}(S, x)$, then x is an isolated point of $E \cap S$ (and therefore the set of all such points is at most countable, and hence \mathcal{H}^1 -null).



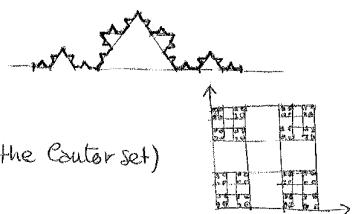
(9)

(iv) The same conclusion as in Remark (iii) holds if μ is absolutely cont. w.r.t. the restriction of \mathcal{H}^k to S . And indeed even if S is a K -dimensional rectifiable set (note we only defined 1-dimensional rectifiable sets).

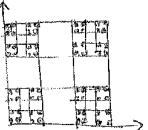
(v) If μ is supported on a purely unrectifiable set E (that is, $\mu(\mathbb{R}^m \setminus E) = 0$) then $V(\mu, x) = \{0\}$ for μ -a.e. x . This follows by the fact that \mathcal{F}_μ contains no non-trivial family (μ_t) , i.e., if $(\mu_t) \in \mathcal{F}_\mu$ then $\mu_t = 0$ for a.e. t , that is, $\mathcal{H}^1(E_t) = 0$. Indeed since $\mu(\mathbb{R}^m \setminus E) = 0$ and $\int \mu_t dt \ll \mu$, then $0 = \int \mu_t(\mathbb{R}^m \setminus E) dt = \int \mathcal{H}^1(E_t \setminus E) dt$ which implies that for a.e. t , E_t is contained in E (up to an \mathcal{H}^1 -null subset). But every rectifiable set contained in a purely unrectifiable set must be \mathcal{H}^1 -null.

(vi) Many popular examples of fractals are purely unrectifiable, and therefore any measure μ supported on them satisfies $V(\mu, x) = \{0\}$ for μ -a.e. by Remark (v). Such examples include:

the Von Koch snowflake

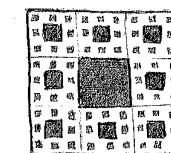


the "Cantor dust," (square of the Cantor set)



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Unlike the previous sets, the "Sierpinski carpet" is not purely unrectifiable.



The "Sierpinski carpet" is the big square minus the union of the gray (open) squares.

However one can prove that the canonical probability measure μ on the Sierp. carpet (the one that you would naturally construct, or more precisely, the one that respects the self-similar structure of the set) is supported on a purely unrectifiable set of the Sierp. Carpet and therefore $V(\mu, x) = \{0\}$ for μ -a.e. x .

Similar conclusions hold for the "Sierpinski gasket," and the "Menger sponge,".

(vii) The structure of the decomposability bundle of a measure μ is completely clarified by Remarks (i) and (v): if $\mu \ll \mathcal{E}'$ then $V(\mu, x) = \mathbb{R}$ for μ -a.e. x and if μ is singular (w.r.t. \mathcal{E}') then $V(\mu, x) = \{0\}$ for μ -a.e. x because μ is supported on a null set (that is, a purely unrectifiable set!).

(And you may guess what happens if μ has both an absolutely continuous and a singular part.)

(viii) The situation is a bit more complicated in dimension $m > 1$. For instance, the following

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question naturally arises:

Is there any singular measure μ (singular w.r.t. \mathcal{L}^m) such that $V(\mu, x) = \mathbb{R}^m$ for μ -a.e. x ?
or equivalently (because of Theorem 1 above),
Is there any singular measure μ such that Rademacher theorem holds in the usual form, that is, every Lipschitz function f is differentiable μ -a.e.?

The answer is NO (for both questions).

For $m=2$ this is proved in Theorem 2 below.
(It was originally proved, in a completely different language, by myself [Rank-one property for derivatives of functions with bounded variation, Proc. Royal Soc. Edinb. A, 123 (1993), 239–274]; it also follows from a result in [ACP]; the proof below is actually a third one.)

For arbitrary m this is a consequence of a result by M. Csörnyei and P. Jones [CJ].

(ix) In the previous remarks we discussed for which measures μ Rademacher theorem holds in the usual form (full-differentiability μ -a.e.) and for which doesn't. One may then ask for which sets E in \mathbb{R}^m there exists a Lipschitz function $f: \mathbb{R}^m \rightarrow \mathbb{R}$.

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which is not differentiable at every point of E .

The answer to this question turns out to be much more delicate. In particular it varies if instead of functions we consider maps $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$, and it actually depends on the dimension m of the target space, as well.

Many people contributed to this problem besides those already mentioned above, but this is another story.

(x) It can be proved that if $\mu \ll \int \mu_t dt$ where each μ_t is the restriction of \mathcal{H}^k to some k -dimensional rectifiable set E_t , then $\dim(V(\mu, x)) \geq k$ for μ -a.e. x .

It is then natural to ask if the converse is true. Note that the answer is yes for trivial reasons when $k=1$, and yes also when $k=m$ (see remark (viii)). Recently András Maté gave an example (with $k=2$ and $m=3$) showing that in general the answer is negative.

We can now pass to the last result of this section.

Theorem 2

If μ is a measure on \mathbb{R}^2 and $V(\mu, x) = \mathbb{R}^2$ for μ -a.e. x , then $\mu \ll \mathcal{L}^2$.

Remark The statement can actually be made more precise: if $A := \{x : V(\mu, x) = \mathbb{R}^2\}$, then the

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restriction of μ to A is absolutely cont. w.r.t. \mathcal{L}^2 .

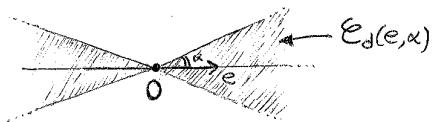
Proof

We assume by contradiction that $\mu \not\ll \mathcal{L}^2$ and $V(\mu, x) = \mathbb{R}^2$ for μ -a.e. x .

The rest of the proof is divided in several steps; most of them look reasonable and we will not prove them, even though you should be aware that in some cases a rigorous proof may be technically delicate (mostly because of measurability issues). We will instead focus on the 5th step, which contains the core of the proof.

Step 1 Possibly replacing μ with its singular part w.r.t. \mathcal{L}^2 , we can assume that μ is singular (recall the "locality principle" in Remark (ii) above).

Given a unit vector e and an angle $\alpha \in (0, \pi/2)$ we denote by $E_d(e, \alpha)$ the (closed, two-sided) cone in the picture



Step 2 Since $V(\mu, x) = \mathbb{R}^2$ for μ -a.e. x , we can find two families (μ_t^i) and (μ_t^2) in \mathcal{F}_μ and two cones $\mathcal{E}^1, \mathcal{E}^2$ such that

- $\mathcal{E}^1 \cap \mathcal{E}^2 = \{0\}$;
- $\text{Tan}(E_t^i, x) \subset \mathcal{E}^i$ for μ_t^i -a.e. x and a.e.t., $i=1,2$;

(14)

- setting $\mu^i := \int \mu_t^i dt$ for $i=1,2$, we have that μ^1 and μ^2 are not mutually singular.

Since we already assumed that $\mu_1, \mu_2 \ll \mu$, the last requirement means that there exists a Borel set A with $\mu(A) > 0$ such that $\mathbb{I}_A \mu \ll \mu_1, \mu_2$.

restriction of μ to A .

The proof of this Step is not difficult but quite delicate.

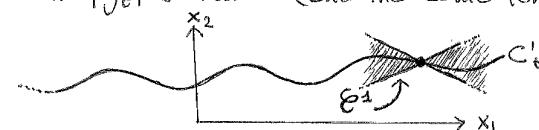
Step 3 We can further assume that $\mathcal{E}^i = E_d(e_i, \alpha)$ where $e_1 := (1, 0)$, $e_2 := (0, 1)$ and $0 < \alpha < \frac{\pi}{2}$ (so $\mathcal{E}^1 \cap \mathcal{E}^2 = \{0\}$).

We should first show that we can assume that the cones \mathcal{E}^1 and \mathcal{E}^2 are "well-separated", that is, the angle between the axes e_1 and e_2 is much larger than α_1 and α_2 . Then we reduce to the case

$e_1 = (1, 0)$, $e_2 = (0, 2)$, $\alpha_1 = \alpha_2$ by little more than a change of variable.

Step 4 We can further assume that each rectifiable set E_t^i is contained in a curve C_t^i (closed and with no boundary) such that $\text{Tan}(C_t^i, x) \subset \mathcal{E}^i$ for every $x \in C_t^i$, every t , and $i=1,2$.

Thus C_t^1 is a graph $x_2 = g_t^1(x_1)$ with $g_t^1: \mathbb{R} \rightarrow \mathbb{R}$ of class \mathcal{E}^1 and s.t. $|g_t^1| \leq \tan \alpha$ (and the same for C_t^2 ...)



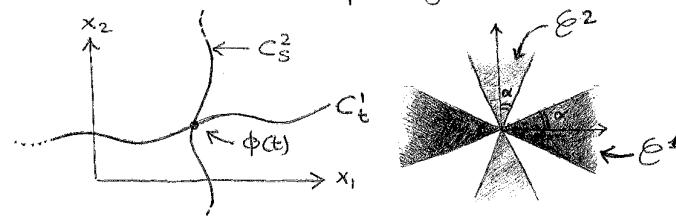
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Step 5 Let $\mu^t := \int_I \mu_t^t dt$ as in step 2.

Then the Radon-Nikodym density of μ^t w.r.t. \mathcal{L}^2 , $\frac{d\mu^t}{d\mathcal{L}^2}(x)$, is finite for \mathcal{H}^1 -a.e. $x \in C_s^2$ and every s .

This is the key step in the proof.

Note that because of the assumptions on the curves C_t^1 and C_s^2 in Step 4, for every $t \in I$ the intersection $C_t^1 \cap C_s^2$ consists of exactly one point x ; we denote by $\phi : I \rightarrow C_s^2$ the map that takes each t in the corresponding x .



(Recall that $\text{Tan}(C_t^1, x) \subset E^2 \quad \forall x \in C_t^1, \forall t$, and $\text{Tan}(C_s^2, x) \subset E^2 \quad \forall x \in C_s^2$.)

We also denote by λ the measure on C_s^2 given by the pushforward of the measure dt on I according to ϕ .

Now we fix $\bar{x} \in C_s^2$ and estimate $\mu^t(B(\bar{x}, r))$ (here $B(\bar{x}, r)$ is the closed ball with center \bar{x} and radius r):

denoting by F the set of all $t \in I$ s.t. $B(\bar{x}, r) \cap C_t^1 \neq \emptyset$ we get :

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$$\mu^t(B(\bar{x}, r)) := \int_{t \in I} \mu_t^t(B(\bar{x}, r)) dt$$

$$\text{recall that } \mu_t^t = \mathbb{1}_{E_t^t}, \mathcal{H}^1 \mapsto \int_{t \in I} \mathcal{H}^1(B(\bar{x}, r) \cap E_t^t) dt$$

$$\text{recall that } E_t^t \subset C_t^1 \mapsto \leq \int_{t \in F} \mathcal{H}^1(B(\bar{x}, r) \cap C_t^1) dt$$

since C_t^1 is the graph of $g_t^1 : \mathbb{R} \rightarrow \mathbb{R}$

and $|g_t^1| \leq \tan \alpha$, then

$$\mathcal{H}^1(B(\bar{x}, r) \cap C_t^1) \leq \int \sqrt{1 + |g_t^1|^2} dt$$

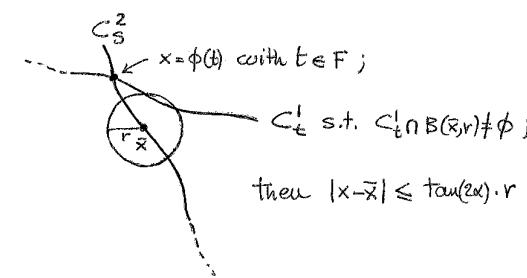
$$\leq \sqrt{1 + \tan^2 \alpha} \cdot r = \frac{2}{\cos \alpha} \cdot r$$

$$\leq c r |F| = c r \lambda(F)$$

↑ constant depending only on α
(we use the same letter for all such constants).

Now $\phi(F)$ is the set of all $x \in C_s^2$ given by the intersections of C_s^2 with all curves C_t^1 that intersect $B(\bar{x}, r)$.

We claim that $\phi(F) \subset B(\bar{x}, \bar{c}r)$ with $\bar{c} = \tan(2\alpha)$



$$\text{then } |x - \bar{x}| \leq \tan(2\alpha) \cdot r.$$

Therefore

$$\mu^t(B(\bar{x}, r)) \leq c r \cdot \lambda(B(\bar{x}, \bar{c}r)),$$

and then, denoting by $\tilde{\lambda}$ the length measure on C_s^2 and $r' := \bar{c}r$

$$\frac{\mu^t(B(\bar{x}, r))}{\mathcal{L}^2(B(\bar{x}, r))} \leq \frac{c r \lambda(B(\bar{x}, \bar{c}r))}{\pi r'^2} = c \frac{\lambda(B(\bar{x}, r))}{r'}$$

$$\leq c \frac{\lambda(B(\bar{x}, r'))}{\mathcal{H}^1(B(\bar{x}, r) \cap C_s^2)} \leq c \frac{\lambda(B(\bar{x}, r'))}{\tilde{\lambda}(B(\bar{x}, r))},$$

(17)

and taking the limit as $r \rightarrow 0$ we finally get

$$\frac{d\mu^1}{d\mathcal{E}^2}(x) \leq \frac{d\lambda}{d\lambda}(x)$$

and the last density is finite (by the Radon-Nikodým theorem) for $\tilde{\lambda}$ -a.e. x , that is, for \mathcal{H}^1 -a.e. $x \in C_s^2$.

Step 6 $\frac{d\mu^1}{d\mathcal{E}^2}(x) < +\infty$ for μ^2 -a.e. x

This follows immediately from Step 5 and the fact that

$\mu^2 := \int_s \mu^2 ds$ and $\mu_s^2 = \text{restriction of } \mathcal{H}^1 \text{ to } E_s^2$, subset of C_s^2 .

Step 7 $\frac{d\mu^1}{d\mathcal{E}^2}(x) < +\infty$ for μ -a.e. $x \in A$, where A comes from Step 2.

This follows from the fact that the restriction of μ to A is absolutely continuous w.r.t. μ^2 (See Step 2).

Step 8 $\frac{d\mu}{d\mathcal{E}^2}(x) < +\infty$ for μ -a.e. $x \in A$.

This follows from Step 7 once we show that $\frac{d\mu}{d\mu^2}(x) < +\infty$ for μ -a.e. $x \in A$; or equivalently that $\frac{d\mu^1}{d\mu}(x) > 0$ for μ -a.e. $x \in A$, which in turn follows from the fact that $1_p \cdot \mu \ll \mu^1$ (Step 2).

Completion of the proof

Since μ is singular w.r.t. \mathcal{E}^2 we have that $\frac{d\mathcal{E}^2}{d\mu}(x) = 0$ for μ -a.e. x , that is, $\frac{d\mu}{d\mathcal{E}^2}(x) = +\infty$ for μ -a.e. x .

This contradicts Step 8 and the fact that $\mu(A) > 0$ (Step 2). \square

3. The Rank-one property for BV functions

I recall the result:

Theorem 3

Let $u: \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a function of class BV, that is, its distributional gradient Du can be represented by a finite measure with values in $m \times n$ matrices.

Then, writing $Du = g \cdot \mu$ with μ positive finite measure and $g: \mathbb{R}^m \rightarrow \mathbb{R}^{m \times n}$ a density in $L^1(\mu)$ s.t. $g \neq 0$ μ -a.e., we have

$$\text{rank}(g(x)) = 1 \text{ for } \mu_s\text{-a.e. } x.$$

\nwarrow singular part of μ .

The proof of this result is an (almost) immediate consequence of the following lemma, which in turn follows from Theorem 2 above:

Lemma

If $u: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a BV function and we write $Du = g \cdot \mu$ as above (now g is a vector-field) then

$$g^\perp(x) \in V(\mu, x) \text{ for } \mu\text{-a.e. } x.$$

\nwarrow rotation of $g(x)$ by 90° (counterclockwise)

Proof

By the coarea formula for BV functions the positive measure $|g| \mu$ can be decomposed as

$$|g| \mu = \int_{-\infty}^{\infty} \mu_t dt$$

(1)

where each μ_t is the restriction of μ to some rectifiable set E_t (roughly speaking, the t -level set of u). Moreover

$$g^\perp(x) \text{ is tangent to } E_t \text{ at } x \text{ for } \mu_t\text{-a.e. } x \text{ and a.e. } t$$

This (and the definition of $V(\mu, x)$) clearly imply the claim. \square

Proof of Theorem 3

Case $m=2$. Let $u = (u_1, \dots, u_n)$ and $Du_i = g_i \cdot \mu$ for every i . By the previous lemma we have that

$$g_i^\perp(x) \in V(\mu, x) \text{ for } \mu\text{-a.e. } x$$

and therefore, recalling that $V(\mu, x) = V(\mu_s, x)$ for μ_s -a.e. x ,

$$g_i(x) \in V^\perp(\mu_s, x) \text{ for } \mu_s\text{-a.e. } x.$$

But since μ_s is singular, Theorem 2 states that $V(\mu_s, x)$ has dimension 1 for μ_s -a.e. x (note that it cannot have dimension 0 because it contains $g_i(x)$ which is $\neq 0$ for some i). Hence the rows $g_i(x)$ of $g(x)$ belong to the same 1-dimensional space (for μ_s -a.e. x) which implies that $\text{rank}(g(x))=1$.

Case $m=3$. The key observation is that given a matrix M , then $\text{rank}(M) \leq 1$ if (and only if) every 2×2 minor of M has rank ≤ 1 .

This suggest that the rank-one property

(2)

(3)

for $u: \mathbb{R}^m \rightarrow \mathbb{R}^n$ actually follows from the rank-one property for the restrictions (traces) of u to (generic) affine planes parallel to the coordinate planes, which has been proved above.

This argument is made rigorous using the slicing formulae for derivatives of BV functions.

□

(2)

4. On the Closability problem

Our main result is the following:

Theorem 4

Let μ be a measure on \mathbb{R}^2 which is not absolutely continuous w.r.t. \mathcal{L}^2 .

Then there exists a sequence of \mathcal{E}' functions f_n on \mathbb{R}^2 with uniformly compact supports and uniformly Lipschitz, such that $f_n \rightarrow 0$ uniformly while $\nabla f_n \rightarrow g \neq 0$ in $L^p(\mu)$ for every $p < +\infty$.

Checking the proof, one easily finds out that it can be extended to higher dimension using the result by H. Csörnyei and P. Jones mentioned before.

To explain the idea of the proof, let us consider the simpler case of a measure μ in \mathbb{R} .

Since $\mu \not\ll \mathcal{L}^1$, we can find a compact set K such that $\mu(K) > 0$ and $\mathcal{L}'(K) = 0$.

Then we can take for every $\delta > 0$ a bounded open set A_δ such that $K \subset A_\delta$ and $\mathcal{L}'(A_\delta) \leq \delta$.

Now we take $f_\delta : \mathbb{R} \rightarrow \mathbb{R}$ given by $f_\delta(-\infty) = 0$ and $f'_\delta := \mathbf{1}_{A_\delta}$, that is

$$f_\delta(x) := \mathcal{L}'((-\infty, x] \cap A_\delta).$$

(1)

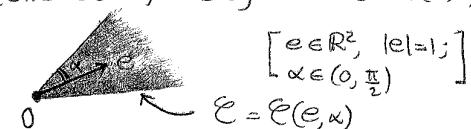
One readily checks that $0 \leq f_\delta \leq \mathcal{L}'(A_\delta) \leq \delta$ (and hence $f_\delta \rightarrow 0$ uniformly as $\delta \rightarrow 0$) and that $f'_\delta = 1$ on K (and therefore $f'_\delta \rightarrow 1$ in $L^p(\mu)$).

The tiny problem is that f_δ is in general Lipschitz but not \mathcal{E}' , but this can be easily fixed by some regularization. (Well, to this end it is essential that $f'_\delta = 1$ on a neighbourhood of K , and not just on K !)

Next, I will try to convince you that "right" requirement on K when we pass to dimension two is that K is directionally null (and not just Lebesgue null).

Definition (directionally null sets)

Let us fix a (one-sided, closed) cone $\mathcal{E} = \mathcal{E}(e, \alpha)$ as in the picture



We then denote by \mathcal{Y} the class of all Lipschitz paths $\gamma : I := [0, 1] \rightarrow \mathbb{R}^2$ such that $\gamma(t) \in \mathcal{E}$ for a.e. t .

Then we say that a Borel set $E \subset \mathbb{R}^2$ is directionally null w.r.t. the cone \mathcal{E} if

$$\mathcal{H}^1(E \cap \gamma(I)) = 0 \quad \text{for every } \gamma \in \mathcal{Y}.$$

More generally, we say that E is directionally δ -small with $\delta > 0$ if

$$\mathcal{H}^1(E \cap \gamma(I)) \leq \delta \quad \text{for every } \gamma \in \mathcal{Y}.$$

(3)

Lemma 5 If K is compact and directionally null (w.r.t. \mathcal{E}) then for every $\delta > 0$ there exists a bounded open set A_δ which is directionally δ -small and contains K .

Actually it suffices to take as A_δ an ϵ -neighbourhood of K with ϵ sufficiently small. That such ϵ exists is a simple compactness argument (but here it is essential that K is compact).

Note that this result is true also without assuming that K is compact, but in this case the proof is highly non-trivial (this result is due to Andras Mathe).

Lemma 6 Let A be a bounded open set ($\in \mathbb{R}^2$) which is directionally δ -small (w.r.t. \mathcal{E}). Then there exists a Lipschitz function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

- (i) $0 \leq \frac{\partial f}{\partial e} \leq 1$ (\mathcal{E}^\perp -) a.e.;
- (ii) $\frac{\partial f}{\partial e} = 1$ a.e. in A ;
- (iii) taking e unitary and orthogonal to e , $|\frac{\partial f}{\partial e}| \leq \frac{1}{\tan \alpha}$
- (iv) $0 \leq f \leq \delta$.

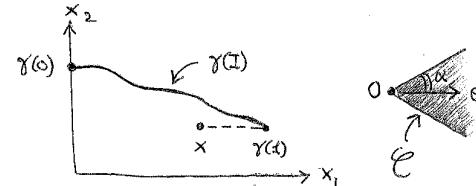
Moreover, as pointed out before, one can use a standard regularization argument to make f of class C^1 .

Proof

We can freely assume that $e = (1, 0)$, $\tilde{e} = (0, 1)$, and A is contained in the half-plane $\{x: x_1 > 0\}$.

(4)

Then for every $x = (x_1, x_2) \in \mathbb{R}^2$ with $x_1 > 0$ we say that $\gamma \in \mathcal{Y}$ is admissible for x if γ is as in the picture:



that is, $(\gamma(0))_1 = 0$, $(\gamma(t))_1 \geq x_1$, $(\gamma(t))_2 = x_2$ (and of course $\dot{\gamma}(t) \in \mathcal{E}$ for a.e.t.).

Then we set

$$f(x) := \sup_{\substack{\gamma \text{ admiss.} \\ \text{for } x}} (H^1(\gamma(I) \cap A) - |\gamma(t) - x|)$$

(and we extend f to the rest of \mathbb{R}^2 setting $f(x) = 0$ if $x_1 < 0$). Then

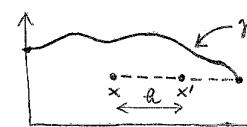
(iv) $f(x) \geq 0 \quad \forall x$ (consider γ —————— \uparrow \rightarrow x); moreover

$f(x) \leq \delta \quad \forall x$ (because A is δ -small), and (iv) is proved.

(i) To prove that $0 \leq \frac{\partial f}{\partial x_i} \leq 1$ we must show that for every x and every $h > 0$ there holds

$$f(x) \leq f(x + he) \leq f(x) + h.$$

Considering γ which is optimal of x' (i.e., realizes the sup in the definition of $f(x')$) and plugging it in the def. of $f(x)$ we get $f(x) \geq f(x') - h$,

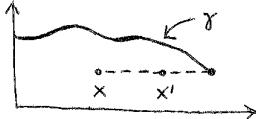


which is the second inequality above.

(5)

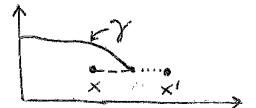
Consider now γ which is optimal for x . There are two possibilities:

a)

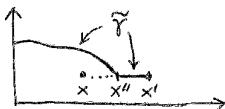


and plugging γ in the definition of $f(x')$ we get $f(x') \geq f(x) + h \geq f(x)$;

b)



in this case let $\tilde{\gamma}$ be



and plugging $\tilde{\gamma}$ in the definition of $f(x')$ we get

$$f(x') \geq \underbrace{f(x) + |x'' - x'|}_{H^1(\gamma(I) \cap A)} + \underbrace{H^1([x'', x'] \cap A)}_{H^1(\tilde{\gamma}(I) \cap A)} \geq f(x).$$

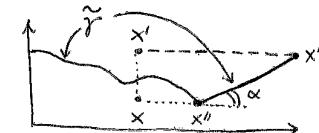
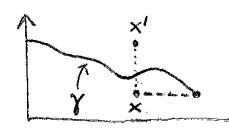
In both cases we get $f(x') \geq f(x)$, which was first inequality in the proof of (i).

- (ii) By examining the proof of (i), cases a) and b) we realize that if the segment $[x, x']$ is contained in A we have $f(x') \geq f(x) + h$, which together with the inequality $f(x) \geq f(x') - h$ implies $f(x') = f(x) + h$. Thus within A the function f is (locally) linear in x , with slope 1 , and in particular $\frac{\partial f}{\partial x} = 1$ a.e. in A .

- (iii) To prove that $\left| \frac{\partial f}{\partial x}(x) \right| \leq \frac{1}{\tan \alpha}$ for a.e. x we must show that given $x \in \mathbb{R}^2$ and $h > 0$ then $\left| f(x+h\tilde{\gamma}) - f(x) \right| \leq \frac{h}{\tan \alpha}$.

(6)

Take indeed γ optimal for x and define $\tilde{\gamma}$ as in the picture:



plugging $\tilde{\gamma}$ in the definition of $f(x')$ we get

$$\begin{aligned} f(x') &\geq H^1(\tilde{\gamma}(I) \cap A) - |x''' - x'| \\ &\geq H^1(\gamma(I) \cap A) - |x''' - x'| \\ &= f(x) + |x'' - x| - |x''' - x'| = f(x) - \frac{h}{\tan \alpha}, \end{aligned}$$

that is, $f(x') - f(x) \geq -\frac{h}{\tan \alpha}$.

And a similar argument gives $f(x') - f(x) \leq \frac{h}{\tan \alpha}$.

□

Lemma 7

Let μ be a measure on \mathbb{R}^2 , and let K be a compact set in \mathbb{R}^2 , directionally null w.r.t. to some cone $C = C(\rho, \alpha)$, and such that $\mu(K) > 0$.

Then there exists a sequence of C^1 functions f_n which are uniformly Lipschitz, converge to 0 uniformly, and $\nabla f_n \rightarrow g \neq 0$ in $L^p(\mu)$

Proof

Using Lemmas 5 and 6 we can easily construct a sequence (\tilde{f}_n) that satisfies all properties except the last one, which is replaced by

$\frac{\partial \tilde{f}_n}{\partial e} = 1$ on K for every n .

(7)

Since the gradients $\nabla \tilde{f}_n$ are uniformly bounded, possibly passing to a subsequence we can assume that $\nabla \tilde{f}_n$ converge weakly* to some g in $L^\infty(\mu)$, and since $\frac{\partial \tilde{f}_n}{\partial e} = 1$ on K , then $g \cdot e = 1$ on K .

In particular $g \neq 0$.

Now, for a given $p < +\infty$ we have that $\nabla \tilde{f}_n$ converge to g weakly in $L^p(\mu)$ and therefore, by a well-known lemma, we can find a sequence f_n of (finite) convex combinations of the functions \tilde{f}_n such that ∇f_n converge strongly to g in $L^p(\mu)$.

And actually ∇f_n will converge strongly to g in $L^p(\mu)$ for every $p < +\infty$. \square

Note that Theorem 3 follows from Lemma 7 together with the next result:

Lemma 8

If μ is a measure on \mathbb{R}^2 which is not absolutely continuous w.r.t. \mathcal{L}^2 , then there exists a compact set K with $\mu(K) > 0$ which is directionally null w.r.t. some cone $C = C(e, \alpha)$.

The key tool in the proof of Lemma 7 is the Rainwater's lemma.

(8)

We can view this result as a generalization of Lebesgue-Radon-Nikodym theorem.

More precisely we state the latter in the form of a dicotomy:

given two measures: μ and λ then one (and only one) of the following statements must hold:

- (i) there exists a λ -null set E such that μ is supported on E . (that is, $\mu(E^c) = 0$);
- (ii) there exists set E with $\lambda(E) > 0$ such that the restriction of λ to E is absolutely cont. w.r.t. μ ($1_E \cdot \lambda \ll \mu$).

(Indeed (i) holds when $\lambda \perp \mu$, while (ii) holds otherwise.)

A similar dicotomy holds also when we replace λ by a family of measures:

Rainwater's lemma

Let μ be a measure (on a compact metric space X) and let \mathcal{F} be weak* compact family of probability measures on X . Then one (and only one) of the following statements holds:

- (i) there exists a Borel set E which is λ -null for every $\lambda \in \mathcal{F}$ such that μ is supported on E ;

(9)

- (iii) there exists a Borel set E and a probability measure σ on \mathcal{F} such that

$$0 \neq \int_{\mathcal{F}} (\mathbf{1}_E \cdot \lambda) d\sigma(\lambda) \ll \mu.$$

Remark This is not the original formulation of Rainwater's lemma, but it can be easily derived from it.

We can now prove Lemma 8 and conclude the proof of Theorem 3.

Proof of Lemma 8

Since $\mu \not\propto \delta^2$, by Theorem 2 the set of all x s.t. $V(\mu, x) \neq \mathbb{R}^2$ must have positive μ -measure. (This is a key step, and here is where we need to be in dimension = 2; for higher dimensions one must use the result by Csörnyei and Jones mentioned before.)

Therefore we can find a cone $\mathcal{C} = \mathcal{C}(e, \alpha)$ and a Borel set F with $\mu(F) > 0$ such that

$$(*) \quad V(\mu, x) \cap \mathcal{C} = \{0\} \text{ for } \mu\text{-a.e. } x \in F.$$

Now we apply Rainwater's lemma to the measure $\tilde{\mu} := \mathbf{1}_F \mu$ and the family \mathcal{F} of all measures λ of the form $\lambda = \text{restriction of } \mathcal{A}^\gamma$ to $\gamma(I)$ with γ path in \mathcal{F} (that is $\gamma: I \rightarrow \mathbb{R}^2$ s.t. $\dot{\gamma}(t) \in \mathcal{C}$ for a.e. t).

(10)

Now, option (ii) in Rainwater's lemma can be ruled out, because it would imply that $V(\mu, x) \supseteq \mathcal{C}$ for all x in a subset of F with positive μ -measure, which would contradict (*).

Thus it remains (i), that is, there exists a Borel set E such that $\mu(F \setminus E) = 0$ (and in particular $\mu(E) > 0$) and E is λ -null for all $\lambda \in \mathcal{F}$, which means exactly that E is directionally null w.r.t. \mathcal{C} . To conclude we just take K a compact subset of E with $\mu(K) > 0$. \square

Remark The (sketch of) proof above is not quite correct, because the measures in \mathcal{F} are not probability measures, and even if they were, \mathcal{F} is not weak* compact. This problem can be easily fixed by taking slightly different measures...

5. Proof of Theorem 1, statement (i)

(1)

Recall the statement:

If μ is a measure on \mathbb{R}^m and $f \in \text{Lip}(\mathbb{R}^m)$, then for μ -a.e. x , f is differentiable at x w.r.t. the subspace $V(\mu, x)$.

As we already pointed out, if $\tilde{\mu} = \int_t \mu dt$ with μ the length measure on some curve C_t , and there exists a vectorfield τ s.t. $\tau(x)$ is tangent to C_t at x for μ_t -a.e. x and a.e.t, then the partial derivative $\frac{\partial f}{\partial \tau}(x)$ exists for $\tilde{\mu}$ -a.e. x .

And actually the same holds if we replace C_t with a rectifiable set E_t .

This observation and the definition of $V(\mu, x)$ suggest that for μ -a.e. x the function f is differentiable at x in a set of directions $D(x)$ that spans $V(\mu, x)$.

This is correct, and not difficult to prove, but it is not enough. We must, indeed prove that:

- (i) f is differentiable at $(\mu\text{-a.e.})x$ in every direction v in $V(\mu, x)$;
- (ii) the partial derivative $\frac{\partial f}{\partial v}(x)$ is linear in v .

This result is achieved thanks to a characterization of $V(\mu, x)$ in terms on normal 1-dimensional currents.

Normal 1-currents (on \mathbb{R}^m)

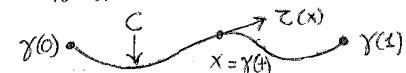
(2)

A normal 1-dimensional current T (on \mathbb{R}^m) is a current with finite mass whose boundary ∂T has finite mass as well.

In practice a 1-current with finite mass is just a vector measure on \mathbb{R}^m with values in \mathbb{R}^m , and it can be represented as $T = \tau \lambda$ with λ a finite positive measure and τ a vectorfield (in $L^1(\lambda)$).

While the boundary of T is the divergence in the sense of distribution, that is $\partial T = \sum_i \frac{\partial}{\partial x_i} (\tau_i \lambda)$, and it has finite mass if it is a (real-valued) measure.

A relevant example of normal 1-current is the following: let C be the image of a simple, Lipschitz path $\gamma: I := [0, 1] \rightarrow \mathbb{R}^m$ (closed or not), and let $T := \tau \lambda$ where λ is the restriction of γ' to C , and $\tau(x)$ is the tangent unit vector given by $\tau(x) := \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|}$ for $x = \gamma(t)$ and a.e.t.



An easy computation of the distributional derivative of T shows that in this case $\partial T = S_{\gamma(1)} - S_{\gamma(0)}$ (a Dirac mass at the end-point minus one at the starting point of C).

This explains the term "boundary".

A decomposition for normal 1-current

We have the following decomposition (a variant—indeed a simplification—of a result by S. Smirnov [Decomposition of solenoidal vector charges..., St. Petersburg Math. J. 5 (1994), 841–867]):

Every normal 1-current $T = \tau\mu$ can be written as

$$(*) \quad T = \int T_t dt$$

where each T_t is the 1-current associated as above to some simple path γ_t ; in particular the positive measure $\mu_t := |T_t|$ is the restriction of H^1 to the rectifiable set $\gamma_t(J)$. Moreover

$$|T| = |\tau|\mu = \int |T_t| dt$$

and $\tau(x) = \tau_t(x)$ for μ_t -a.e. x and a.e.t.

An auxiliary bundle

Let μ be fixed. By the Lebesgue-Radon-Nikodym theorem we can write every normal 1-current T as $T = \tau(\mu + \mu')$ with $\tau \in L^1(\mu + \mu')$ and μ' a measure singular w.r.t. μ . We then denote by \mathcal{H}'_μ the class of these τ (as elements of $L^1(\mu)$) and let $x \mapsto V'(\mu, x)$ be the μ -essential span of \mathcal{H}'_μ , that is, the (unique) minimal element of the class G_μ of all bundles $x \in \mathbb{R}^m \mapsto G(x) \in G(\mu)$ s.t. for every $\tau \in \mathcal{H}'_\mu$ there holds $\tau(x) \in V(x)$ for μ -a.e. x .

Lemma 9 (Characterization of $V(\mu, x)$)

Take μ , $V'(\mu, x)$ as above, and $V(\mu, x)$ as usual.

Then

$$V(\mu, x) = V'(\mu, x) \text{ for } \mu\text{-a.e. } x.$$

(3)

Idea of proof

The proof of this lemma is quite technical, but the main ideas can be easily explained.

To prove the inclusion $V'(\mu, x) \subset V(\mu, x)$ for μ -a.e. x , we must show that for every normal 1-current $T = \tau(\mu + \mu')$ there holds $\tau(x) \in V(\mu, x)$ for μ -a.e. x , and this follows immediately from the decomposition of T given in $(*)$ and the definition of $V(\mu, x)$.

For the inclusion $V(\mu, x) \subset V'(\mu, x)$ for μ -a.e. x we must show that given $\int \mu_t dt \ll \mu$ with μ_t being the restriction of H^1 to some rectifiable set E_t , then $Tan(E_t, x) \subset V'(\mu, x)$ for μ_t -a.e. x and a.e.t.

The idea is to cover each E_t with countably many curves $C_{t,n}$, associate to each $C_{t,n}$ a normal 1-current $T_{t,n}$ as above, and consider the normal 1-current T given by $T = \int \sum_n T_{t,n} dt$ (possibly multiplying each $T_{t,n}$ by some small positive number so to make the sums and the integral converge). Then the idea is that $V'(\mu, x)$ contains the vectorified $\tau_{t,n}(x)$ associated to $T_{t,n}$, which spans $Tan(C_{t,n}, x)$, which in turn agrees with $Tan(E_t, x)$

This is "essentially" correct, but none of these (claimed) inclusions actually hold as it is stated.

□

(4)

Idea of the proof of Theorem 1 (i)

(5)

Let τ be a vectorfield in \mathcal{M}'^μ , that is, the vectorfield associated to a normal \mathbb{L} -current T by the formula $T = \tau(\mu + \mu')$.

Let $T = \int T_t dt$ be the decomposition of T in (*).

Thus $\mu_t := |T_t|$ is the restriction of \mathcal{M}' to a rectifiable set E_t , and τ_t (defined by $T_t = \tau_t \mu_t$) is tangent to E_t μ_t -a.e. and $\tau_t(x) = \frac{\tau(x)}{|\tau(x)|}$ for μ_t -a.e. x and a.e.t.

Then, as already pointed out before, f is differentiable at x in the direction $\tau(x) = \tau_t(x)$ for μ_t -a.e. x and a.e.t. In other words

$$\frac{\partial f}{\partial \tau}(x) \text{ exists for } \underline{\mu\text{-a.e.} x} \quad \xrightarrow{\text{(actually } (\mu+\mu')\text{-a.e.} x)}$$

There are now two key facts:

- the vectorfields τ in \mathcal{M}'^μ span $V(\mu/x)$ (Lemma 9);
- \mathcal{M}'^μ is a vector space.

Using these facts it is not difficult to show that

$$\frac{\partial f}{\partial v}(x) \text{ exists for every } v \in V(\mu/x) \text{ and } \mu\text{-a.e.} x$$

To conclude the proof of statement (i) of Theorem 1

we need to show that $\frac{\partial f}{\partial v}(x)$ is linear in v (for μ -a.e. x).

To prove this fact we use that for every

the partial derivative $\frac{\partial f}{\partial \tau}$ is characterized by

the following formula for the boundary (distributional divergence) of the normal \mathbb{L} -current $fT = f\tau(\mu + \mu')$:

$$\partial(fT) = f \cdot \partial T + \frac{\partial f}{\partial \tau}(\mu + \mu')$$

This means that $\frac{\partial f}{\partial \tau}$ is the Radon-Nikodym density of the measure $\partial(fT) - f \cdot \partial T$ w.r.t. $\mu + \mu'$ (or μ , if we are only interested on the values of $\frac{\partial f}{\partial \tau}$ for μ -a.e. point).

The key point is that this formula is linear in T ,

And from this we can derive the linearity of $\frac{\partial f}{\partial v}(x)$ in v for μ -a.e. x .

□

6. Proof of Theorem 1(ii)

Recall the statement :

given a measure μ on \mathbb{R}^m there exists a Lipschitz function f on \mathbb{R}^m such that, for μ -a.e. x ,

$$\frac{\partial f}{\partial v}(x) \text{ does not exist for every } v \notin V(\mu, x).$$

The construction of f for general m is quite involved. We start therefore with a construction in dimension $m=1$.

Proposition 10

Let E be a Borel null set in \mathbb{R} .

Then there exists a Lipschitz function f on \mathbb{R} which is not differentiable at every $x \in E$.

In particular, by choosing E so that it supports the singular measure μ we obtain that f is not differentiable at μ -a.e. x .

Remark This result is known since a long time, cf. [Z. Zahorski, Bull. Soc. Math. France 74 (1946), 167-178].

Proof

We assume for simplicity that E is compact (the general case is slightly more complicated).

Using that E is Lebesgue-null we find a sequence of open sets A_n such that :

(1)

- (i) the sets A_n are bounded and $A_0 \supset A_1 \supset A_2 \supset \dots \supset E$;
- (ii) each A_n has finitely many connected components (here we use that E is compact);
- (iii) for every n and every J connected component of A_n there holds $\mathcal{L}^1(A_{n+1}) \leq 2^{-n} \mathcal{L}^1(J)$ (here we need that A_n has finitely many conn. comp.).

(2)

Now we let $g_n : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $g_n(-\infty) = 0$ and $g_n = \mathbf{1}_{A_n}$, that is

$$g_n(x) := \mathcal{L}^1((-\infty, x] \cap A_n);$$

and set

$$f_n := \sum_{k=0}^{\infty} (-1)^k g_k; \quad f = \lim_{n \rightarrow \infty} f_n.$$

(It can be easily checked that the functions f_n are 1-Lipschitz and converge uniformly to $f = \sum_{k=0}^{\infty} (-1)^k g_k$, which is therefore also 1-Lipschitz.)

Fix now $x \in E$ and for every n let $J_n = (a_n, b_n)$ be the connected component of A_n that contains x , thus $a_n \nearrow x$ and $b_n \searrow x$.

Now, if n is even then $f'_n = 1$ on A_n , and in particular f_n is linear with slope 1 on J_n . Thus

$$1 = \frac{f_n(b_n) - f_n(a_n)}{b_n - a_n} \approx \frac{f(b_n) - f(a_n)}{b_n - a_n}$$

Indeed condition (iii) ensures that $\|f - f_n\|_{\infty} \leq 2^{1-n}(b_n - a_n)$.

(4)

Since $a_n \nearrow x$ and $b_n \searrow x$, from this estimate we can infer that

$$D^+f(x) := \limsup_{\substack{h \rightarrow 0 \\ \uparrow}} \frac{f(x+h)-f(x)}{h} = 1$$

upper derivative of f at x .

On the other hand, when n is odd then f_n has slope 0 on J_n , and arguing as before we get

$$D^-f(x) := \liminf_{\substack{h \rightarrow 0 \\ \uparrow}} \frac{f(x+h)-f(x)}{h} = 0.$$

lower derivative

Thus f is not differentiable at x . \square

(3)

The extension of the previous construction to higher dimension is quite complicated.

On the other hand, the fact that f is obtained by successive perturbations suggests that this construction may be recast as a Banach-Mazur game, or more generally that f could be obtained by some category argument, which would hopefully grant some simplification (in particular in view of the extension to higher dimension).

Among all possible approaches (all work!) we have chosen one suggested some years ago by Bernd Kirchheim.

Assume now that μ is supported in $[0,1]$, and let X be the space of 1-lipschitz functions $f : [0,1] \rightarrow \mathbb{R}$, endowed with the supremum distance.

Let then Y be the subset of all $f \in X$ such that

$$D^+f(x) = +1 \text{ and } D^-f(x) = -1 \text{ for } \mu\text{-a.e. } x.$$

Proposition 11

Y is residual in X , and in particular it is not empty (and every f in Y is clearly not differentiable at μ -a.e. point).

Proof

For every $s > 0$ and every $f \in X$ we let $T_s f$ be the function of $[0,1]$ defined by

$$T_s f(x) := \sup_{|h| \geq s} \frac{f(x+h)-f(x)}{h}.$$

Step 1 The map $T_s : X \rightarrow L^1(\mu)$ is continuous for every $s > 0$, and converge pointwise to $T_0 : X \rightarrow L^1(\mu)$. (I omit the verification).

Thus T_0 is a Baire-one map, and therefore the set Z of all $f \in X$ such that $T_0 f$ is continuous is residual in X .

Step 2 If $T_0 f$ is continuous at f then $T_0 f = f$ μ -a.e.

Assume by contradiction that this is not the case, and choose any sequence of smooth $f_n \in X$ that

(5)

converge to f .

We can then find a compact null set K with $\mu(K) > 0$ and $\varepsilon > 0$ such that $T_0 f_n(x), T_0 f(x) < 1 - \varepsilon \quad \forall x \in K$.

This implies in particular that $f'_n(x) < 1 - \varepsilon \quad \forall x \in K$.

We can then choose an open set A_n such that $A_n \supset K$, $f'_n(x) < 1 - \varepsilon \quad \forall x \in A_n$, and $\mathcal{L}^1(A_n) \rightarrow 0$ as $n \rightarrow \infty$.

Next we let $g_n : [0,1] \rightarrow \mathbb{R}$ be the function defined by $g_n(0) = 0$, $g'_n = \mathbf{1}_{A_n}$, and set

$$\tilde{f}_n := f_n + \varepsilon g_n.$$

Then $\|g_n\|_\infty = \mathcal{L}^1(A_n) \rightarrow 0$, and therefore

$\tilde{f}_n \rightarrow f$ uniformly.

Moreover one easily checks that \tilde{f}'_n is still 1-Lipschitz (essentially because $g'_n = 0$ out of A_n and in A_n $\tilde{f}'_n = f'_n + \varepsilon g'_n \leq 1 - \varepsilon + \varepsilon \leq 1$), and finally

$$T_0 \tilde{f}_n(x) \geq T_0 f_n(x) + \varepsilon \quad \forall x \in K.$$

But this inequality implies that $T_0 \tilde{f}_n$ does not converge in $L^1(\mu)$ to $T_0 f$, which contradicts the assumption that T_0 is continuous at f .

Step 3 To conclude the proof just check that $T_0 f(x) = \ell$ implies that $D^+ f(x) = \ell$.

So every f in the residual set Z (the point of

(6)

continuity of T_0) satisfies (by Step 2) $D^+ f = \ell$ μ -a.e. But then also the set of all f such that $D^- f = -\ell$ μ -a.e. is residual (by a similar argument). And since Y is the intersection of these two residual sets, it is also residual. \square

The strategy used to prove Proposition 4 has been used in [AM] to prove Theorem 1(ii).

In dimension $m > 1$, however, the proof is a bit more delicate than above, and even the definition of the space X is rather involved.

Therefore I will not give the details here.

Let me just point out, however, that the "right" property of the set K in this higher dimensional generalization, is not being \mathcal{L}^m -null but being directionally null (with respect to suitable cones).

And that the perturbations g_n constructed from K are exactly the functions constructed in Lemma 6 for the solution of the closability problem.