A_1 theory of weights for singular integral operators

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1 Introduction: motivation

• **The A_p theorem**

In the celebrated paper [26], B. Muckenhoupt characterized the class of weights for which the Hardy–Littlewood maximal operator is bounded on $L^p(w)$; the surprisingly simple necessary and sufficient condition is the celebrated A_p condition of Muckenhoupt, namely $\|M\|_{L^p(w)}$ is finite if and only if the quantity

$$[w]_{A_p} := \sup_Q \left(\fint_Q w(x)\, dx \right) \left(\fint_Q w(x)^{1-p'}\, dx \right)^{p-1} \tag{1.1}$$

is finite. We will call it the A_p constant of the weight although some authors call it the "characteristic" or the "norm" of the weight. The operator norm $\|M\|_{L^p(w)}$ will depend somehow upon the A_p constant of w, but the first result expressing this dependency was proved by S. Buckley [2] as part of his Ph.D. thesis:

> Let $1 < p < \infty$ and let $w \in A_p$, then the Hardy-Littlewwod maximal function satisfies the following operator estimate:
>
> $$\|M\|_{L^p(w)} \leq c_n\, p'\, [w]_{A_p}^{\frac{1}{p-1}} \tag{1.2}$$
>
> namely,
>
> $$\sup_{w \in A_p} \frac{1}{[w]_{A_p}^{\frac{1}{p-1}}} \|M\|_{L^p(w)} \leq c_n\, p'.$$

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Furthermore the result is sharp in the sense that: for any $\epsilon > 0$

$$\sup_{w\in A_p} \frac{1}{[w]_{A_p}^{\frac{1}{p-1}-\epsilon}} \|M\|_{L^p(w)} = \infty \tag{1.3}$$

Buckley showed with an specific example in (1.2) that the exponent is optimal but the explanation why this exponent is the correct one is due to the following fact:

$$\|M\|_{L^p(\mathbb{R}^n)} \approx \frac{1}{p-1} \qquad p \to 1$$

as shown in [24] as part of a general phenomena.

• **The Fefferman-Stein inequality**

More or less at the same time the following inequality was proved by C. Fefferman and E. Stein [10] for the Hardy-Littlewood maximal function:

$$\|Mf\|_{L^{1,\infty}(w)} \leq c \int_{\mathbb{R}^n} |f|\, Mw\, dx, \tag{1.4}$$

The inequality (1.4) is interesting on its own because it is an improvement of the classical weak-type $(1,1)$ property of the Hardy-Littlewood maximal operator M. However, the crucial new point of view is that it can be seen as a sort of duality for M since the following L^p inequality as a consequence:

$$\int_{\mathbb{R}^n} (Mf)^p\, w\, dx \leq c_p \int_{\mathbb{R}^n} |f|^p\, Mw\, dx \qquad f, w \geq 0. \tag{1.5}$$

This estimate follows from the classical interpolation theorem of Marcinkiewicz although a direct proof can be given avoiding interpolation. Both results (1.4) and (1.5) were proved by C. Fefferman and E.M. Stein in [10] to derive the following vector-valued extension of the classical Hardy-Littlewood maximal theorem: for every $1 < p, q < \infty$, there is a finite constant $c = c_{p,q}$ such that

$$\left\| \Big(\sum_j (Mf_j)^q \Big)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} \leq c \left\| \Big(\sum_j |f_j|^q \Big)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)}. \tag{1.6}$$

This is a very deep theorem and has been used a lot in modern harmonic analysis explaining the central role of inequality (1.4). Nevertheless, th

• **The A_1 condition**

The A_p condition defined above makes sense when $p \in (1, \infty)$, however there are two extreme cases A_1 and A_∞ which play a central role in the theory.

The A_1 class of weights can be defined immediately from Fefferman-Stein's inequality (1.4) and in fact this is what the authors did in the paper: if the weight $w \in A_1$ then

$$\|Mf\|_{L^{1,\infty}(w)} \leq c_n\, [w]_{A_1} \int_{\mathbb{R}^n} |f|\, w dx.$$

The A_1 condition can be read directly from Fefferman-Stein's inequality (1.4) and in fact was already introduced by these authors in that paper: the weight w is an A_1 weight or satisfies the A_1 condition if there is a finite constant c such that

$$Mw \leq c\, w \quad \text{a.e.} \tag{1.7}$$

As before, we denote by $[w]_{A_1}$ the smallest of these constants c. Then if $w \in A_1$

$$\|Mf\|_{L^{1,\infty}(w)} \leq c_n\, [w]_{A_1} \int_{\mathbb{R}^n} |f|\, w dx.$$

This follows from (1.4) but also it follows from (1.5) that

$$\|M\|_{L^p(w)} \leq c_n\, p'\, [w]_{A_1}^{1/p}$$

We pint out that the A_1 condition can be defined from (1.1) by letting p goes down to 1.

• **The A_∞ condition**

To define the A_∞ class of weights the key observation is that the A_p condition is decreasing on p:

$$[w]_{A_q} \leq [w]_{A_p} \qquad 1 \leq p < q$$

implying that $A_p \subset A_q$. Hence it is natural to define

$$A_\infty := \cup_{p>1} A_p.$$

The A_∞ class of weights shares a lot of interesting properties, we remit to [7] for more information. The problem is that this definition does not lead to any appropriate constant. It is known that another possible way of defining a constant for this class is by means of the quantity

$$[w]_{A_\infty}^{exp} := \sup_Q \fint_Q w\, \exp\left(\fint_Q -\log w\right),$$

as can be found in for instance in [13]. This constant was introduced by Hruščev in [14]. This definition is natural because is obtained letting $p \to \infty$ in the definition of the A_p constant. In fact we have by Jensen's inequality:

$$[w]_{A_\infty}^{exp} \leq [w]_{A_p}.$$

On the other hand in [16] the authors use a "new" A_∞ constant (which was originally introduced by Fujii in [11] and rediscovered later by Wilson in [36]) which is more suitable.

Definition 1.1.
$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q)\, dx.$$

Observe that $[w]_{A_\infty} \geq 1$ by the Lebesgue differentiation theorem.

It is not difficult to show using the logarithmic maximal function

$$M_0 f := \sup_Q \exp\left(\fint_Q \log|f| \right) \chi_Q$$

that

$$[w]_{A_\infty} \leq c_n\, [w]_{A_\infty}^{exp}$$

See [16] for details. In fact it is shown in the same paper with explicit examples that $[w]_{A_\infty}$ is much smaller than $[w]_{A_\infty}^{exp}$ (actually exponentially smaller).

We also refer the reader to the forthcoming work of Duoandikoetxea, Martin-Reyes and Ombrosi [8] for a discussion regarding different definitions of A_∞ classes (see also [7]).

• **Improving the A_p theorem: the mixed $A_p - A_\infty$ approach**

As was mentioned above the exponent in Buckley's estimate (1.2) cannot be improved, however that estimate can be really improved in a different way.

Theorem 1.1. *Let $1 < p < \infty$ and let $\sigma = w^{-1/(p-1)}$, then*

$$\|M\|_{L^p(w)} \leq c_n p' \big([w]_{A_p} [\sigma]_{A_\infty}\big)^{1/p}. \tag{1.8}$$

Using the duality relationship

$$[\sigma]_{A_{p'}}^{\frac{1}{p'}} = [w]_{A_p}^{\frac{1}{p}}$$

we see immediately that

$$\big([w]_{A_p} [\sigma]_{A_\infty}\big)^{1/p} \leq [w]_{A_p}^{\frac{1}{p-1}}$$

yielding (1.2).

This result and Theorem 1.2 below were first proved in [16] with a simplified proof in [18]. A key estimate was the following result.

Theorem 1.2 (An optimal reverse Hölder inequality)**.** *Define $r_w := 1 + \dfrac{1}{c_n\, [w]_{A_\infty}}$, where c_n is a dimensional constant. Note that $r'_w \approx [w]_{A_\infty}$.*

a) If $w \in A_\infty$, then

$$\left(\fint_Q w^{r_w} \right)^{1/r_w} \leq 2 \fint_Q w.$$

b) Furthermore, the result is optimal up to a dimensional factor: If a weight w satisfies the RHI, i.e., there exists a constant K such that

$$\left(\fint_Q w^r\right)^{1/r} \leq K \fint_Q w,$$

then there exists a dimensional constant $c = c_n$, such that $[w]_{A_\infty} \leq c_n \, K \, r'$.

We mention that some similar one dimensional results have been independently obtained by O. Beznosova and A. Reznikov in [1] by means of the Bellman function technique.

The original proof of this result can be found in [16] but it was simplified an improved in [18] (it can be found as well in Paseky lectures notes [31]). Also it should be mentioned that this reverse Hölder property was completely avoided in [32] to derive the mixed $A_p - A_\infty$ result (1.8).

The following corollary which is usually called the *open* property of the A_p condition, is an important consequence.

Corollary 1.1 (The precise open property)**.** *Let $1 < p < \infty$ and let $w \in A_p$. Then $w \in A_{p-\epsilon}$ where*

$$\epsilon = \frac{p-1}{r(\sigma)'} = \frac{p-1}{1 + \tau_n [\sigma]_{A_\infty}}$$

where as usual $\sigma = w^{1-p'}$. Furthermore

$$[w]_{A_{p-\epsilon}} \leq 2^{p-1} \, [w]_{A_p}$$

Proof. Since $w \in A_p$, $\sigma \in A_{p'} \subset A_\infty$, and hence

$$\fint_Q w \left(\fint_Q \sigma^{r(\sigma)}\right)^{\frac{p-1}{r(\sigma)}} \leq \fint_Q w \left(2 \fint_Q \sigma\right)^{p-1}.$$

Choose ϵ so that $\frac{p-1}{r(\sigma)} = p - \epsilon - 1$, namely $\epsilon = \frac{p-1}{r(\sigma)'}$ and observe that $\epsilon > 0$ and $p - \epsilon > 1$. This yields that $w \in A_{p-\epsilon}$.

$\square$

A previous already similar result was obtained for A_1 weights in [22] that played an important role in that and subsequent papers. Later on an interesting extension to A_p weights, where the exponent r_w is $r_w' \approx \|M\|_{L^{p'}(\sigma)}$, was further obtained in [21].

• **Factorization.**

Muckenhoupt already observed in [26] that it follows from the definition of the A_1 class of weights that if $w_1, w_2 \in A_1$, then the weight

$$w = w_1 w_2^{1-p}$$

is an A_p weight. Furthermore, with our definitions, it follows easily

$$[w]_{A_p} \leq [w_1]_{A_1}[w_2]_{A_1}^{p-1} \tag{1.9}$$

He conjectured that any A_p weight can be factored out in this fashion. This conjecture was proved by P. Jones's showing that if $w \in A_p$ then there are A_1 weights w_1, w_2 such that $w = w_1 w_2^{1-p}$. It is also well known that the modern approach to this question uses completely different path and it is due to J. L. Rubio de Francia as can be found in [13] and also in [7]. In these notes we will use the "easy" part of the theorem and more precisely the estimate (1.9).

• **Two weight problem: sharp Sawyer's theorem**

Another highlight of the theory of weights is the two weight characterization of the maximal theorem due to E. Sawyer [35] (see also [13]):

> Let $1 < p < \infty$, and let u, σ two unrelated weights, then there is a finite constant C such that
>
> $$\|M(f\sigma)\|_{L^p(u)} \leq C\,\|f\|_{L^p(\sigma)} \tag{1.10}$$
>
> if and only if there is a finite constant K such that for any cube Q
>
> $$\left(\int_Q M(\sigma\,\chi_Q)^p\,udx\right)^{1/p} \leq K\,\sigma(Q)^{1/p} < \infty$$

K. Moen proved in [25] a quantitative version of Sawyer's theorem as follows: if we let

$$[u,\sigma]_{S_p} = \sup_Q \frac{\left(\int_Q M(\sigma\,\chi_Q)^p\,udx\right)^{1/p}}{\sigma(Q)^{1/p}}$$

then we have.

Theorem 1.3. *Let $1 < p < \infty$ and let u, σ and $\|M\|$ as above. Then*

$$[u,\sigma]_{S_p} \leq \|M\| \leq c_n\,p'\,[u,\sigma]_{S_p}$$

In a recent joint work with E. Rela [32] an application of this result has been found obtaining an improvement of the mixed $A_p - A_\infty$ Theorem 1.1 as well as some other two weight estimates with *"bumps"* conditions providing new quantitative estimates of results derived in [29].

2 Singular Integrals: The A_1 theory

Very recently, the so called Muckenhoupt-Wheeden conjecture has been disproved by Reguera-Thiele in [34]. This conjecture claimed that there exists a constant c such that for any function f and any weight w

$$\|Hf\|_{L^{1,\infty}(w)} \leq c \int_{\mathbb{R}} |f|\, Mw dx. \tag{2.1}$$

where H is the Hilbert transform. The failure of the conjecture was previously obtained by M.C. Reguera in [33] for a special model operator T instead of H. This conjecture was motivated by C. Fefferman and E. Stein inequality (1.4) for the Hardy-Littlewood maximal function.

That this conjecture was believed to be false was already mentioned in [28] where the best positive result in this direction so far can be found, and where M is replaced by $M_{L(\log L)^\epsilon}$, i.e., a maximal type operator that is "ϵ-logarithmically" bigger than M:

$$\|Tf\|_{L^{1,\infty}(w)} \leq c_{T,\epsilon} \int_{\mathbb{R}^n} |f|\, M_{L(\log L)^\epsilon}(w) dx \qquad w \geq 0.$$

where T is the Calderón-Zygmund operator T. Until very recently the constant of the estimate did not play any essential role except, perhaps, for the fact that it blows up. If we check the computations in [28] we find that $c_\epsilon \approx e^{\frac{1}{\epsilon}}$. It turns out that improving this constant would lead to understanding interesting questions in the area. Recently this estimate has been improved in [17] where the exponential blow up $e^{\frac{1}{\epsilon}}$ has been reduced to a linear blow up $\frac{1}{\epsilon}$. A second improvement consists of replacing T by the maximal singular integral operator T^*. The method in [28] cannot be used directly since the linearity of T played a crucial role in the argument.

Theorem 2.1. *Let T be a Calderón-Zygmund operator with maximal singular integral operator T^*. Then for any $0 < \epsilon \leq 1$,*

$$\|T^* f\|_{L^{1,\infty}(w)} \lesssim \frac{c_T}{\epsilon} \int_{\mathbb{R}^n} |f(x)|\, M_{L(\log L)^\epsilon}(w)(x)\, dx \qquad w \geq 0 \tag{2.2}$$

It seems that the right conjecture is the following:

$$\|T^* f\|_{L^{1,\infty}(w)} \leq c_T \int_{\mathbb{R}^n} |f(x)|\, M_{L \log\log L}(w)(x)\, dx \qquad w \geq 0. \tag{2.3}$$

We remark that the operator $M_{L(\log L)^\epsilon}$ is pointwise smaller than $M_r = M_{L^r}$, $r > 1$ where $M_r(w) = M(w^r)^{1/r}$.

In these lectures we will present a baby version of this result which can be essentially found in [23] and [22] with the improvements obtained in [16]. We will provide some hints on how to prove Theorem 2.1 together with L^p versions of that in Section 6.

To prove this result we have to study first the corresponding weighted $L^p(w)$ estimates with $1 < p < \infty$ and $w \in A_1$ being the result this time fully sharp. The final part of the proofs of both theorem can be found in Sections 4 and 5 and are essentially . Again we remit to [30] for a more complete discussion about these estimates and their variants.

We state now the main theorems of this chapter. From now on T will always denote any Calderón-Zygmund operator and we assume that the reader is familiar with the classical unweighted theory.

Our goal is to prove the following results.

Theorem 2.2. *Let T be a Calderón–Zygmund operator and let $1 < p < \infty$. Then for any weight w and $r > 1$,*

$$\|Tf\|_{L^p(w)} \leq c_T pp'(r')^{\frac{1}{p'}} ||f||_{L^p(M_r w)}. \tag{2.4}$$

Now, using the sharp exponent in the reverse Hölder inequality for weights in the A_∞ class from Theorem 1.2 the following result follows easily.

Corollary 2.1. *Let T be a Calderón–Zygmund operator and let $1 < p < \infty$. Then if $w \in A_\infty$ we obtain*

$$\|Tf\|_{L^p(w)} \leq c_T pp'[w]_{A_\infty}^{1/p'} ||f||_{L^p(Mw)}, \tag{2.5}$$

and if $w \in A_1$,

$$\|Tf\|_{L^p(w)} \leq c_T pp'[w]_{A_\infty}^{1/p'} [w]_{A_1}^{1/p} ||f||_{L^p(w)} \tag{2.6}$$

and hence

$$\|T\|_{L^p(w)} \leq c_T \, pp' \, [w]_{A_1}. \tag{2.7}$$

The exponent in inequality (2.7) is best possible similarly as in (1.2).

A very nice application of a very especial extrapolation theorem obtained by J. Duoandikoetxea in [6] can be deduced from (2.7).

Corollary 2.2. *Under the same assumption as before we have, if $1 \leq q < p$*

$$\|T\|_{L^p(w)} \leq c_{T,p,q} \, [w]_{A_q}.$$

As an application of (2.4) we obtain the following endpoint estimate.

Theorem 2.3. *Let T be a Calderón–Zygmund operator. Then for any weight w and $r > 1$,*

$$||Tf||_{L^{1,\infty}(w)} \leq c_T \log{(e + r')} ||f||_{L^1(M_r w)}. \tag{2.8}$$

Now, using again the reverse Hölder inequality for weights from Theorem 1.2 we get the following consequence.

Corollary 2.3. *[The logarithmic growth theorem]*
Let T be a Calderón–Zygmund operator. Then

1. *If $w \in A_\infty$*

$$||Tf||_{L^{1,\infty}(w)} \leq C \log{(e+[w]_{A_\infty})} ||f||_{L^1(Mw)}.$$

2. *If $w \in A_1$*

$$||Tf||_{L^{1,\infty}(w)} \leq c_T [w]_{A_1} \log{(e+[w]_{A_\infty})} ||f||_{L^1(w)}.$$

We remark that these theorems can be further improved by replacing T by T^*, the maximal singular integral operator. Again, the method presented now cannot be applied because is based on the fact that T is linear while T^* is not. See [17].

In view of [27] this could be the best possible result, namely $[w]_{A_1} \log{(e+[w]_{A_\infty})}$ cannot be replaced by $[w]_{A_1} \log{(e+[w]_{A_\infty})}^\alpha$, with $0 \leq \alpha < 1$. This came as a big surprise.

3 Estimates involving A_∞ weights

In Harmonic Analysis, there are a number of important inequalities of the form

$$\int_{\mathbb{R}^n} |Tf(x)|^p \, w(x)\, dx \leq C \int_{\mathbb{R}^n} |Sf(x)|^p \, w(x)\, dx, \tag{3.1}$$

where T and S are operators. Typically, T is an operator with some degree of singularity (e.g., a singular integral operator), S is an operator which is, in principle, easier to handle (e.g., a maximal operator), and w is in some class of weights.

The standard technique for proving such results is the so-called good-λ inequality of Burkholder and Gundy. These inequalities compare the relative measure of the level sets of S and T: for every $\lambda > 0$ and $\epsilon > 0$ small,

$$w(\{y \in \mathbb{R}^n : |Tf(y)| > 2\,\lambda, |Sf(y)| \leq \lambda\epsilon\}) \leq C\,\epsilon\, w(\{y \in \mathbb{R}^n : |Sf(y)| > \lambda\}). \tag{3.2}$$

Here, the weight w is usually assumed to be in the class $A_\infty = \cup_{p>1} A_p$. Given inequality (3.2), it is easy to prove the strong-type inequality (3.1) for any p, $0 < p < \infty$, as well as the corresponding weak-type inequality

$$\|Tf\|_{L^{p,\infty}(w)} \leq C\,\|Sf\|_{L^{p,\infty}(w)}. \tag{3.3}$$

In these notes the special case of

$$\|Tf\|_{L^p(w)} \leq c\,\|Mf\|_{L^p(w)} \tag{3.4}$$

where T is a Calderón-Zygmund operator and M is the maximal function, will play a central role in the proof of Theorem 2.2. Estimate (3.4) was proved by Coifman-Fefferman in the celebrated paper [3]. In our context the weight w will also satisfy the A_∞ condition but the problem is that the behavior of the constant is too rough. We need a more precise result for very specific weights.

Lemma 3.1. *Let w be any weight and let $1 \leq p, r < \infty$. Then, there is a constant $c = c(n, T)$ such that:*

$$\|Tf\|_{L^p(M_r w)^{1-p})} \leq \, cp\, \|Mf\|_{L^p(M_r w)^{1-p})}$$

This is the main improvement in [23] of [22] where we had obtained logarithmic growth on p. It is an important step towards the proof of the Theorem 2.2.

The above mentioned good λ of Coifman-Fefferman is not sharp because instead of $c\,p$ gives $C(p) \approx 2^p$ because

$$[(M_r w)^{1-p})]_{A_p} \approx (r')^{p-1}$$

The proof of this lemma is tricky and it combines another variation the of Rubio de Francia algorithm together with a sharp L^1 version of (3.4):

$$\|Tf\|_{L^1(w)} \leq c[w]_{A_q}\|Mf\|_{L^1(w)} \qquad w \in A_q,\ 1 \leq q < \infty$$

The original proof given in [23] of this estimate was based on an idea of R. Fefferman-Pipher from [9] which combines a sharp version of the good-λ inequality of S. Buckley together with a sharp reverse Hölder property of the weights. The result of Buckley establishes a very interesting exponential improvement of the good-λ estimate of above mentioned Coifman-Fefferman estimate as can be found in [2]:

$$|\{x \in \mathbb{R}^n : T^*(f) > 2\lambda, Mf < \gamma\lambda\}| \leq c_1 e^{-c_2/\gamma}|\{T^*(f) > \lambda\}| \qquad \lambda, \gamma > 0 \tag{3.5}$$

where T^* is the maximal singular integral operator. New more flexible arguments can be found in [17] leading to the following.

Lemma 3.2. *Let $w \in A_\infty$. Then*

$$\|Tf\|_{L^1(w)} \leq c\,[w]_{A_\infty}\,\|Mf\|_{L^1(w)}.$$

We now finish this section by proving the "tricky" Lemma 3.1. The proof is based on the following lemma which is another variation of the Rubio de Francia algorithm.

Lemma 3.3. *Let* $1 < s < \infty$ *and let* w *be a weight. Then there exists a nonnegative sublinear operator* R *satisfying the following properties:*
(a) $h \leq R(h)$
(b) $\|R(h)\|_{L^s(w)} \leq 2\|h\|_{L^s(w)}$
(c) $R(h)w^{1/s} \in A_1$ *with*

$$[R(h)w^{1/s}]_{A_1} \leq cs'$$

Proof. We consider the operator

$$S(f) = \frac{M(f\,w^{1/s})}{w^{1/s}}$$

Since $\|M\|_{L^s} \sim s'$, we have

$$\|S(f)\|_{L^s(w)} \leq cs'\|f\|_{L^s(w)}.$$

Now, define the Rubio de Francia operator R by

$$R(h) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{S^k(h)}{(\|S\|_{L^s(w)})^k}.$$

It is very simple to check that R satisfies the required properties. □

Proof of Lemma 3.1. We are now ready to give the proof of the "tricky" Lemma, namely to prove

$$\left\|\frac{Tf}{M_r w}\right\|_{L^p(M_r w)} \leq cp \left\|\frac{Mf}{M_r w}\right\|_{L^p(M_r w)}$$

By duality we have,

$$\left\|\frac{Tf}{M_r w}\right\|_{L^p(M_r w)} = |\int_{\mathbb{R}^n} Tf\,h\,dx| \leq \int_{\mathbb{R}^n} |Tf|\,h\,dx$$

for some $\|h\|_{L^{p'}(M_r w)} = 1$. By Lemma 3.3 with $s = p'$ and $v = M_r w$ there exists an operator R such that
(A) $h \leq R(h)$
(B) $\|R(h)\|_{L^{p'}(M_r w)} \leq 2\|h\|_{L^{p'}(M_r w)}$
(C) $[R(h)(M_r w)^{1/p'}]_{A_1} \leq cp$.

We want to make use of property (C) combined with the following two facts: First, if $w_1, w_2 \in A_1$, and $w = w_1 w_2^{1-p} \in A_p$, then by (1.9)

$$[w]_{A_p} \leq [w_1]_{A_1}[w_2]_{A_1}^{p-1}$$

Second, if $r > 1$ then $(Mf)^{\frac{1}{r}} \in A_1$ by the Coifman-Rochberg theorem from [4] but we need a more precise estimate which follows from the proof:

$$[(Mf)^{\frac{1}{r}}]_{A_1} \leq c_n\,r'.$$

Hence combining we obtain

$$\begin{aligned}[R(h)]_{A_3} &= [R(h)(M_r w)^{1/p'}\big((M_r w)^{1/2p'}\big)^{-2}]_{A_3}\\ &\leq [R(h)(M_r w)^{1/p'}]_{A_1}[(M_r w)^{1/2p'}]^2_{A_1}\\ &\leq cp.\end{aligned}$$

Therefore, by Lemma 3.2 and by properties (A) and (B),

$$\begin{aligned}\int_{\mathbb{R}^n}|Tf|h\,dx &\leq \int_{\mathbb{R}^n}|Tf|R(h)\,dx\\ &\leq c[R(h)]_{A_3}\int_{\mathbb{R}^n}M(f)R(h)\,dx\\ &\leq cp\left\|\frac{Mf}{M_r w}\right\|_{L^p(M_r w)}\|h\|_{L^{p'}(M_r w)}.\end{aligned}$$

□

4 Proof of Theorem 2.2

We will prove

$$\|Tf\|_{L^{p'}(M_r w)^{1-p'})} \leq cp'\,(r')^{1-1/pr}\|f\|_{L^{p'}(w^{1-p'})}$$

from which (2.4) follows since $t^{1/t}\leq 2$, $t\geq 1$.

We consider the equivalent dual estimate:

$$\|T^t f\|_{L^{p'}(M_r w)^{1-p'})} \leq cp'\,(r')^{1-1/pr}\|f\|_{L^{p'}(w^{1-p'})}$$

Then use the "tricky" Lemma 3.1 since T^t is also a Calderón-Zygmund operator

$$\|\frac{T^t f}{M_r w}\|_{L^{p'}(M_r w))} \leq p'\,c\,\|\frac{Mf}{M_r w}\|_{L^{p'}(M_r w))}$$

We could use now Theorem 1.3 but we use a more direct method. Indeed, by Hölder's inequality with exponent pr,

$$\frac{1}{|Q|}\int_Q fw^{-1/p}w^{1/p} \leq \left(\frac{1}{|Q|}\int_Q w^r\right)^{1/pr}\left(\frac{1}{|Q|}\int_Q (fw^{-1/p})^{(pr)'}\right)^{1/(pr)'}$$

and hence,

$$(Mf)^{p'} \leq (M_r w)^{p'-1}M\Big((fw^{-1/p})^{(pr)'}\Big)^{p'/(pr)'}$$

From this, and by the classical unweighted maximal theorem with the sharp constant,

$$\begin{aligned}\left\|\frac{Mf}{M_r w}\right\|_{L^{p'}(M_r w)} &\leq c\Big(\frac{p'}{p'-(pr)'}\Big)^{1/(pr)'}\left\|\frac{f}{w}\right\|_{L^{p'}(w)}\\ &= c\Big(\frac{rp-1}{r-1}\Big)^{1-1/pr}\left\|\frac{f}{w}\right\|_{L^{p'}(w)}\\ &\leq cp\,(r')^{1-1/pr}\left\|\frac{f}{w}\right\|_{L^{p'}(w)}.\end{aligned}$$

5 Proof of Theorem 2.3

The proof is based on initial ideas from [28]. Applying the Calderón-Zygmund decomposition to f at level λ, we get a family of pairwise disjoint cubes $\{Q_j\}$ such that

$$\lambda < \frac{1}{|Q_j|}\int_{Q_j}|f| \leq 2^n\lambda$$

Let $\Omega = \cup_j Q_j$ and $\widetilde{\Omega} = \cup_j 2Q_j$. The "good part" is defined by

$$g = \sum_j f_{Q_j}\chi_{Q_j}(x) + f(x)\chi_{\Omega^c}(x)$$

and the "bad part" b as

$$b = \sum_j b_j$$

where

$$b_j(x) = (f(x) - f_{Q_j})\chi_{Q_j}(x)$$

Then, $f = g + b$.

However, it turns out that b is "excellent" and g is really "ugly". It is so good the b part that we obtain the maximal function on the right hand side:

$$w\{x\in(\widetilde{\Omega})^c : |Tb(x)| > \lambda\} \leq \frac{c}{\lambda}\int_{\mathbb{R}^n}|f|Mw\,dx$$

by a well known argument using the cancellation of the b_j and that we omit. Also the term $w(\widetilde{\Omega})$ is the level set of the maximal function and the Fefferman-Stein applies (again we obtain the maximal function on the right hand side).

Combining we have

$$\begin{aligned}w\{x\in\mathbb{R}^n : |Tf(x)| > \lambda\} \leq\, & w(\widetilde{\Omega}) + w\{x\in(\widetilde{\Omega})^c : |Tb(x)| > \lambda/2\}\\ &+ w\{x\in(\widetilde{\Omega})^c : |Tg(x)| > \lambda/2\}.\end{aligned}$$

and the first two terms are already controlled:

$$w(\widetilde{\Omega}) + w\{x \in (\widetilde{\Omega})^c : |Tb(x)| > \lambda/2\} \leq \frac{c}{\lambda} \int_{\mathbb{R}^n} |f|\, Mw dx.$$

Now, by Chebyschev and Theorem 2.2: if $p > 1$ and $r > 1$ we have

$$\begin{aligned}
&\lambda\, w\{x \in (\widetilde{\Omega})^c : |Tg(x)| > \lambda/2\} \\
&\leq \lambda\, c_T\, (p')^p (r')^{p-1} \frac{1}{\lambda^p} \int_{\mathbb{R}^n} |g|^p M_r(w\chi_{(\widetilde{\Omega})^c}) dx \\
&\leq c_T\, (pp')^p (r')^{p-1} \int_{\mathbb{R}^n} |g|\, M_r(w\chi_{(\widetilde{\Omega})^c}) dx.
\end{aligned}$$

By known standard geometric arguments we have

$$\int_{\mathbb{R}^n} |g| M_r(w\chi_{(\widetilde{\Omega})^c}) dx \leq c_n \int_{\mathbb{R}^n} |f| M_r w dx.$$

Now if we choose $p = 1 + \frac{1}{\log(e+r')}$ we evan continue with

$$\leq c_T\, \log(r') \int_{\mathbb{R}^n} |f| M_r w dx \qquad r > 1.$$

This estimate combined with the previous one completes the proof.

6 Proof/discussion of Theorem 2.1

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