Reduction principles for Sobolev type embeddings

Andrea Cianchi
Dipartimento di Matematica e Informatica "U.Dini"
Università di Firenze
Piazza Ghiberti 27, 50122 Firenze (Italy)
e-mail: cianchi@unifi.it

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1 Rearrangements

We recall in the present Section and in the next one some basic definitions and properties on rearrangements and rearrangement-invariant spaces. We refer to [BS] for a comprehensive treatment of these topics.

Let $(\mathcal{R}, \nu)$ be a σ-finite, non-atomic measure space. In most of our applications, $\mathcal{R}$ will just be an open subset of $\mathbb{R}^n$, $n \geq 1$, and ν the Lebesgue measure.
We define

$$\mathcal{M}(\mathcal{R}, \nu) = \mathcal{M}(\mathcal{R}) = \{u : \mathcal{R} \to [-\infty, \infty] \,:\, u \text{ is } \nu\text{-measurable}\}.$$

We also set

$$\mathcal{M}_+(\mathcal{R}, \nu) = \mathcal{M}_+(\mathcal{R}) = \{u \in \mathcal{M}(\mathcal{R}) : u \geq 0\},$$

and

$$\mathcal{M}_0(\mathcal{R}, \nu) = \mathcal{M}_0(\mathcal{R}) = \{u \in \mathcal{M}(\mathcal{R}) : u \text{ is finite a.e. in } \mathcal{R}\}.$$

Definition 1.1 [Distribution function] Let $u \in \mathcal{M}(\mathcal{R})$. The distribution function $\mu_u : [0, \infty) \to [0, \infty]$ of u is defined as

$$\mu_u(t) = \nu(\{x \in \mathcal{R} : |u(x)| > t\}) \qquad \text{for } t \geq 0. \tag{1.1}$$

Definition 1.2 [Decreasing rearrangement] Let $u \in \mathcal{M}(\mathcal{R})$. The decreasing rearrangement $u^* : [0, \infty) \to [0, \infty]$ of u is defined as

$$u^*(s) = \sup\{t \geq 0 : \mu_u(t) > s\} \qquad \text{for } s \in [0, \infty), \tag{1.2}$$

(with the convention that $\sup \emptyset = 0$).

Note that u^* is the generalized right-continuous inverse of μ_u.

One has that $u^*(s) = 0$ if $s \geq \nu(\mathcal{R})$.

Definition 1.3 [Equimeasurable functions] Two functions $u \in \mathcal{M}(\mathcal{R}_1, \nu_1)$ and $v \in \mathcal{M}(\mathcal{R}_2, \nu_2)$ are said to be equimeasurable, or equidistributed, if $\mu_u(t) = \mu_v(t)$ for $t \geq 0$.

Remark 1.4 Equation (1.2) entails that

$$\mu_{u^*}(t) = \mu_u(t) \qquad \text{for } t \geq 0. \tag{1.3}$$

Indeed,

$$\{u^* > t\} = [0, \mu_u(t)).$$

Thus, u and u^* are equimeasurable functions. In particular, u^* is the unique non-increasing right-continuous function in $[0, \infty)$ equimeasurable with u.

Assume that $\nu(\mathcal{R}) < \infty$. Then the signed decreasing rearrangement $u^\circ : [0, \nu(\mathcal{R})] \to [-\infty, \infty]$ is given by

$$u^\circ(s) = \sup\{t \in \mathbb{R} : \nu(\{u > t\}) > s\} \qquad \text{for } s \in [0, \nu(\mathcal{R})]. \tag{1.4}$$

The function u° is equimeasurable with u. Moreover, on setting

$$u_+ = \frac{|u| + u}{2} \quad \text{and} \quad u_- = \frac{|u| - u}{2},$$

the positive and negative parts of u, respectively, one has that

$$u^\circ(s) = u_+^*(s) - u_-^*(\nu(\mathcal{R}) - s) \qquad \text{for a.e. } s \in [0, \nu(\mathcal{R})]. \tag{1.5}$$

Proposition 1.5 [Properties of rearrangements] Let $u, v \in \mathcal{M}_0(\mathcal{R})$, $\{u_k\} \subset \mathcal{M}_0(\mathcal{R})$, and $\lambda \in \mathbb{R}$. Then:
(i) If $|v| \leq |u|$ a.e., then $v^* \leq u^*$.
(ii) $(\lambda u)^* = |\lambda| u^*$.
(iii) $(u + v)^*(s_1 + s_2) \leq u^*(s_1) + v^*(s_2) \quad$ for $s_1, s_2 \geq 0$.
(iv) If $|u_n| \nearrow |u|$ a.e., then $u_n^* \nearrow u^*$.
(v) $u^*(\mu_u(t)) \leq t \quad$ and $\quad u^*(\mu_u(t)^-) \geq t \quad$ if $\mu_u(t) < \infty$.
(vi) $\mu_u(u^*(s)) \leq s \quad$ and $\mu_u(u^*(s)^-) \geq s \quad$ if $u^*(s) < \infty$.
(vii) If $\phi : [0, \infty] \to [0, \infty]$ is increasing and continuous, then

$$\phi(|u|)^* = \phi(u^*). \tag{1.6}$$

Proposition 1.6 [Invariance of integrals] *Let $u \in \mathcal{M}(\mathcal{R})$, and let $\phi : [0, \infty] \to [0, \infty]$ be non-decreasing. Then*

$$\int_{\mathcal{R}} \phi(|u(x)|)\, dx = \int_0^\infty \phi(u^*(s))\, ds \tag{1.7}$$

and

$$\operatorname{ess\,sup}_{x \in \mathcal{R}} |u(x)| = \sup_{s \geq 0} u^*(s) = u^*(0). \tag{1.8}$$

Proof. Let us assume, for simplicity, that ϕ is left-continuous, the general case requiring just few modifications. We have that

$$\phi(t) = \phi(0) + \int_{[0,t)} d\phi(\tau) \qquad \text{for } t \geq 0.$$

Thus, by Fubini's theorem,

$$\begin{aligned} \int_{\mathcal{R}} \phi(|u(x)|)\, d\nu(x) &= \int_{\mathcal{R}} \left(\phi(0) + \int_{[0,|u(x)|)} d\phi(\tau)\right) d\nu(x) = \\ &= \phi(0)\nu(\mathcal{R}) + \int_0^\infty \left(\int_{\mathcal{R}} \chi_{[0,|u(x)|)}(\tau) d\nu(x)\right) d\phi(\tau) \\ &= \phi(0)\nu(\mathcal{R}) + \int_0^\infty \mu_u(\tau) d\phi(\tau). \end{aligned} \tag{1.9}$$

An analogous chain with u replaced with u^* yields

$$\int_0^\infty \phi(u^*(s))\, ds = \phi(0)\nu(\mathcal{R}) + \int_0^\infty \mu_{u^*}(\tau) d\phi(\tau).$$

Hence, equation (1.7) follows, since $\mu_u = \mu_{u^*}$.
By the latter equation,

$$\operatorname{ess\,sup}_{x\in\mathcal{R}} |u(x)| = \sup_{s\geq 0} u^*(s) = u^*(0),$$

whence (1.8) follows. □

Corollary 1.7 [Invariance of L^p norms] *Let $p \in (0,\infty]$ and let $u \in L^p(\mathcal{R})$. Then*

$$\|u\|_{L^p(\mathcal{R})} = \|u^*\|_{L^p(0,\infty)}. \tag{1.10}$$

Theorem 1.8 [Hardy-Littlewood inequality] *Let $u, v \in \mathcal{M}(\mathcal{R})$. Then*

$$\int_{\mathcal{R}} |u(x)v(x)|\, d\nu(x) \leq \int_0^\infty u^*(s)v^*(s)\, ds\,. \tag{1.11}$$

Proof. Denote by χ_E the characteristic function of a set E. We have that

$$|u(x)| = \int_0^\infty \chi_{\{|u|>t\}}(x) dt \qquad \text{for } x \in \mathcal{R}, \tag{1.12}$$

and, similarly,

$$u^*(s) = \int_0^\infty \chi_{\{u^*>t\}}(s) dt \qquad \text{for } s \geq 0. \tag{1.13}$$

From (1.12)- (1.13) and their analogues for v and v^* we obtain, via Fubini's theorem,

$$\begin{aligned} \int_{\mathcal{R}} |u(x)v(x)|\, d\nu(x) &= \int_{\mathcal{R}} \left(\int_0^\infty \chi_{\{|u|>t\}}(x) dt\right)\left(\int_0^\infty \chi_{\{|v|>\tau\}}(x) d\tau\right) d\nu(x) \\ &= \int_0^\infty \int_0^\infty \left(\int_{\mathcal{R}} \chi_{\{|u|>t\}}(x)\chi_{\{|v|>\tau\}}(x) d\nu(x)\right) d\tau dt \\ &= \int_0^\infty \int_0^\infty \left(\int_{\mathcal{R}} \chi_{\{|u|>t\}\cap\{|v|>\tau\}}(x) d\nu(x)\right) d\tau dt \\ &= \int_0^\infty \int_0^\infty \nu(\{|u|>t\}\cap\{|v|>\tau\}) d\tau dt\,, \end{aligned} \tag{1.14}$$

and

$$\int_0^\infty u^*(s)v^*(s)\,ds = \int_0^\infty \left(\int_0^\infty \chi_{\{u^*>t\}}(s)dt\right)\left(\int_0^\infty \chi_{\{v^*>\tau\}}(s)d\tau\right)ds \tag{1.15}$$
$$= \int_0^\infty \int_0^\infty \left(\int_0^\infty \chi_{\{u^*>t\}}(s)\chi_{\{v^*>\tau\}}ds\right)d\tau dt$$
$$= \int_0^\infty \int_0^\infty \min\{\mu_u(t), \mu_v(\tau)\}d\tau dt\,.$$

Since

$$\nu(\{|u| > t\} \cap \{|v| > \tau\}) \leq \min\{\mu_u(t), \mu_v(\tau)\} \quad \text{for } t, \tau > 0,$$

inequality (1.11) follows from (1.14) and (1.15). □

Definition 1.9 [Maximal function of the rearrangement] Given $u \in \mathcal{M}(\mathcal{R})$, the function $u^{**} : (0,\infty) \to [0,\infty]$ is defined as

$$u^{**}(s) = \frac{1}{s}\int_0^s u^*(r)\,dr \quad \text{for } s > 0. \tag{1.16}$$

Proposition 1.10 *Let $u \in \mathcal{M}(\mathcal{R})$. Then*

$$u^{**}(s) = \frac{1}{s}\sup\left\{\int_E |u(x)|d\nu(x) : |E| = s\right\} \quad \textit{for } s > 0. \tag{1.17}$$

Proposition 1.11 *Let $u, v \in \mathcal{M}_0(\mathcal{R})$. Then*

$$(u+v)^{**}(s) \leq u^{**}(s) + v^{**}(s) \qquad \textit{for } s > 0. \tag{1.18}$$

2 Rearrangement invariant spaces

Let $L \in (0,\infty]$. We say that a functional $\|\cdot\|_{X(0,L)} : \mathcal{M}_+(0,L) \to [0,\infty]$ is a *function norm*, if, for all f, g and $\{f_j\}_{j\in\mathbb{N}}$ in $\mathcal{M}_+(0,L)$, and all $\lambda \geq 0$, the following properties hold:

(P1) $\|f\|_{X(0,L)} = 0$ if and only if $f = 0$; $\|\lambda f\|_{X(0,L)} = \lambda\|f\|_{X(0,L)}$;
$\|f+g\|_{X(0,L)} \leq \|f\|_{X(0,L)} + \|g\|_{X(0,L)}$;

(P2) $f \leq g$ a.e. implies $\|f\|_{X(0,L)} \leq \|g\|_{X(0,L)}$;

(P3) $f_j \nearrow f$ a.e. implies $\|f_j\|_{X(0,L)} \nearrow \|f\|_{X(0,L)}$;

(P4) $\|\chi_E\|_{X(0,L)} < \infty$ if $E \subset (0,L)$, with $|E| < \infty$;

(P5) $\int_E f(s)\,ds \leq C\|f\|_{X(0,L)}$ if $E \subset (0,L)$ with $|E| < \infty$, for some constant $C = C(X,E)$.

If, in addition,

(P6) $\qquad \|f\|_{X(0,L)} = \|g\|_{X(0,L)}$ whenever $f^* = g^*$,

we say that $\|\cdot\|_{X(0,L)}$ is a *rearrangement-invariant function norm.*

With any rearrangement-invariant function norm $\|\cdot\|_{X(0,L)}$, it is associated another functional on $\mathcal{M}_+(0,L)$, denoted by $\|\cdot\|_{X'(0,L)}$, and defined, for $g \in \mathcal{M}_+(0,L)$, as

$$\|g\|_{X'(0,L)} = \sup_{\substack{f\geq 0 \\ \|f\|_{X(0,L)}\leq 1}} \int_0^L f(s)g(s)\,ds.$$

It turns out that $\|\cdot\|_{X'(0,L)}$ is also a rearrangement-invariant function norm, which is called the *associate function norm* of $\|\cdot\|_{X(0,L)}$.

Given a rearrangement-invariant function norm $\|\cdot\|_{X(0,\nu(\mathcal{R}))}$, the rearrangement-invariant space $X(\mathcal{R},\nu)$ is defined as the collection of all functions $u \in \mathcal{M}(\mathcal{R},\nu)$ such that the expression

$$\|u\|_{X(\mathcal{R},\nu)} = \|u^*\|_{X(0,\nu(\mathcal{R}))} \tag{2.1}$$

is finite.

Theorem 2.1 *The functional $\|\cdot\|_{X(\mathcal{R},\nu)}$ is actually a norm, under which $X(\mathcal{R},\nu)$ is complete. Hence, $X(\mathcal{R},\nu)$ is a Banach space. Moreover, $X(\mathcal{R},\nu) \subset \mathcal{M}_0(\mathcal{R},\nu)$ for any rearrangement-invariant space $X(\mathcal{R},\nu)$.*

If $\mathcal{R} = (0,L)$ for some $L \in (0,\infty]$, and ν is the Lebesgue measure, we denote $X(\mathcal{R},\nu)$ simply by $X(0,L)$.

The space $X(0,\nu(\mathcal{R}))$ is called the *representation space* of $X(\mathcal{R},\nu)$.

If $\mathcal{R}$ is a subset of $\mathbb{R}^n$, we also denote by $X_{\mathrm{loc}}(\mathcal{R},\nu)$ the space of all functions $u \in \mathcal{M}(\mathcal{R},\nu)$ such that $u\chi_G \in X(\mathcal{R},\nu)$ for every compact set $G \subset \mathcal{R}$.

The rearrangement-invariant space $X'(\mathcal{R},\nu)$ built upon the function norm $\|\cdot\|_{X'(0,\nu(\mathcal{R}))}$ is called the *associate space* of $X(\mathcal{R},\nu)$. It turns out that $X''(\mathcal{R},\nu) = X(\mathcal{R},\nu)$. Furthermore, the *Hölder type inequality*

$$\int_{\mathcal{R}} |u(x)v(x)|\,d\nu(x) \leq \|u\|_{X(\mathcal{R},\nu)}\|v\|_{X'(\mathcal{R},\nu)} \tag{2.2}$$

holds for every $u \in X(\mathcal{R},\nu)$ and $v \in X'(\mathcal{R},\nu)$.

The notation

$$X(\mathcal{R},\nu) \to Y(\mathcal{R},\nu)$$

means that$X(\mathcal{R},\nu) \subseteq Y(\mathcal{R},\nu)$, and the identity map is a bounded (linear) operator.

For any rearrangement-invariant spaces $X(\mathcal{R},\nu)$ and $Y(\mathcal{R},\nu)$, we have that

$$X(\mathcal{R},\nu) \to Y(\mathcal{R},\nu) \quad \text{if and only if} \quad Y'(\mathcal{R},\nu) \to X'(\mathcal{R},\nu), \tag{2.3}$$

with the same embedding norms.

Given any $\lambda > 0$, the *dilation operator* E_λ, is defined at a function $f \in \mathcal{M}(0,L)$ as

$$(E_\lambda f)(s) = \begin{cases} f(\lambda^{-1}s) & \text{if } 0 < s \leq \lambda L \\ 0 & \text{if } \lambda L < s < L. \end{cases}$$

The operator E_λ is bounded on any rearrangement-invariant space $X(0,L)$, with norm not exceeding $\max\{1,\frac{1}{\lambda}\}$; namely

$$\|E_\lambda f\|_{X(0,L)} \leq \max\{1,\frac{1}{\lambda}\}\|f\|_{X(0,L)}$$

for every $f \in X(0,L)$.

The *Hardy–Littlewood–Pólya principle* asserts that if the functions $u, v \in \mathcal{M}(\mathcal{R},\nu)$ satisfy

$$\int_0^s u^*(r)\,dr \le \int_0^s v^*(r)\,dr \qquad \text{for } s \in (0,\infty),$$

then

$$\|u\|_{X(\mathcal{R},\nu)} \le \|v\|_{X(\mathcal{R},\nu)}$$

for every rearrangement-invariant space $X(\mathcal{R},\nu)$.

Let $X(\mathcal{R},\nu)$ and $Y(\mathcal{R},\nu)$ be rearrangement invariant spaces. Then

$$X(\mathcal{R},\nu) \subset Y(\mathcal{R},\nu) \qquad \text{if and only if} \qquad X(\mathcal{R},\nu) \to Y(\mathcal{R},\nu).$$

If $\nu(\mathcal{R}) < \infty$, then

$$L^\infty(\mathcal{R},\nu) \to X(\mathcal{R},\nu) \to L^1(\mathcal{R},\nu) \tag{2.4}$$

for every rearrangement-invariant space $X(\mathcal{R},\nu)$.

We now recall the definitions of some customary rearrangement-invariant spaces. Throughout, we use the convention that $\frac{1}{\infty} = 0$, and $0 \cdot \infty = 0$.

Lebesgue spaces. A basic instance of a function norm is the standard *Lebesgue r.i. function norm* $\|\cdot\|_{L^p(0,L)}$, for $p \in [1,\infty]$, upon which the Lebesgue spaces $L^p(\mathcal{R},\nu)$ are built.

Lorentz spaces. The *Lorentz spaces* yield an extension of the Lebesgue spaces. Assume that $1 \le p, q \le \infty$. We define the functionals $\|\cdot\|_{L^{p,q}(0,L)}$ and $\|\cdot\|_{L^{(p,q)}(0,L)}$ as

$$\|f\|_{L^{p,q}(0,L)} = \left\| s^{\frac{1}{p}-\frac{1}{q}} f^*(s) \right\|_{L^q(0,L)} \quad \text{and} \quad \|f\|_{L^{(p,q)}(0,L)} = \left\| s^{\frac{1}{p}-\frac{1}{q}} f^{**}(s) \right\|_{L^q(0,L)},$$

respectively, for $f \in \mathcal{M}_+(0,L)$. One can show that

$$\|\cdot\|_{L^{p,q}(0,L)} \approx \|\cdot\|_{L^{(p,q)}(0,L)} \quad \text{if } 1 < p \le \infty\,, \tag{2.5}$$

up to multiplicative constants. The functional $\|\cdot\|_{L^{(p,q)}(0,L)}$ is a rearrangement-invariant function norm. If one of the conditions

$$\begin{cases} 1 < p < \infty,\ 1 \le q \le \infty, \\ p = q = 1, \\ p = q = \infty, \end{cases} \tag{2.6}$$

is satisfied, then $\|\cdot\|_{L^{p,q}(0,L)}$ is equivalent to a rearrangement-invariant function norm. The corresponding rearrangement-invariant spaces $L^{p,q}(\mathcal{R},\nu)$ and $L^{(p,q)}(\mathcal{R},\nu)$ on a measure space $(\mathcal{R},\nu)$ are called *Lorentz spaces.*

Let us recall that

$$L^{p,p}(\mathcal{R},\nu) = L^p(\mathcal{R},\nu) \text{ for every } p \in [1,\infty],$$

and that

$$1 \le q \le r \le \infty \text{ implies } L^{p,q}(\mathcal{R},\nu) \to L^{p,r}(\mathcal{R},\nu),$$

with equality if and only if $q = r$.
Moreover, if

$$\nu(\mathcal{R}) < \infty, \text{ then } L^{p_1,q_1}(\mathcal{R},\nu) \to L^{p_2,q_2}(\mathcal{R},\nu) \text{ if } p_1 > p_2, \text{ for all } q_1 \text{ and } q_2.$$

Orlicz spaces. A generalization of the Lebesgue spaces in a different direction is provided by the Orlicz spaces. Let $A : [0,\infty) \to [0,\infty]$ be a Young function, namely a convex (non trivial), left-continuous function vanishing at 0. Any such function takes the form

$$A(t) = \int_0^t a(\tau)d\tau \qquad \text{for } t \geq 0, \tag{2.7}$$

for some non-decreasing, left-continuous function $a : [0,\infty) \to [0,\infty]$ which is neither identically equal to 0, nor to ∞.
A Young function A is said to dominate another Young function B near infinity if positive constants c and t_0 exist such that

$$B(t) \leq A(ct) \qquad \text{for} \quad t \geq t_0\,. \tag{2.8}$$

The functions A and B are called equivalent near infinity if they dominate each other near infinity.
The *Luxemburg r.i. function norm* built upon A is defined as

$$\|f\|_{L^A(0,L)} = \inf\left\{\lambda > 0 : \int_0^L A\left(\frac{f(s)}{\lambda}\right) ds \leq 1\right\}$$

for $f \in \mathcal{M}_+(0,L)$. The Orlicz space $L^A(\mathcal{R},\nu)$ is the rearrangement-invariant space associated with such function norm.
In particular,

$$L^A(\mathcal{R},\nu) = L^p(\mathcal{R},\nu) \text{ if } A(t) = t^p \text{ for some } p \in [1,\infty),$$

and

$$L^A(\mathcal{R},\nu) = L^\infty(\mathcal{R},\nu) \text{ if } A(t) = \infty\chi_{(1,\infty)}(t).$$

If $\nu(\mathcal{R}) < \infty$, then

$$L^A(\mathcal{R},\nu) \to L^B(\mathcal{R},\nu) \quad \text{if and only if } A \text{ dominates } B \text{ near infinity}\,. \tag{2.9}$$

We denote by $L^p \log^\alpha L(\mathcal{R},\nu)$ the Orlicz space associated with a Young function equivalent to $t^p(\log t)^\alpha$ near infinity, where either $p > 1$ and $\alpha \in \mathbb{R}$, or $p = 1$ and $\alpha \geq 0$. The notation $\exp L^\beta(\mathcal{R},\nu)$ will be used for the Orlicz space built upon a Young function equivalent to e^{t^β} near infinity, where $\beta > 0$. Also, $\exp\exp L^\beta(\mathcal{R},\nu)$ stands for the Orlicz space associated with a Young function equivalent to $e^{e^{t^\beta}}$ near infinity.

3 Perimeter and isoperimetric inequalities

Throughout this Section, Ω denotes an open subset of $\mathbb{R}^n$, with $n \geq 1$. References for the material of this section include [Ma2, Zi].

Definition 3.1 [Sets of finite perimeter] Let E be a Lebesgue measurable set in $\mathbb{R}^n$. The perimeter $P(E;\Omega)$ of E in Ω is defined as

$$P(E;\Omega) = \sup\left\{\int_E \mathrm{div} U\, dx : U \in C_0^\infty(\Omega,\mathbb{R}^n), |U(x)| \leq 1\right\}. \tag{3.1}$$

The set E is said to have finite perimeter in Ω if $P(E;\Omega) < \infty$.

When $\Omega = \mathbb{R}^n$, we denote $P(E, \mathbb{R}^n)$ simply by $P(E)$, and if $P(E) < \infty$, then E is just called a set of finite perimeter.

Note that

$$P(E;\Omega) = P(\Omega \setminus E;\Omega) \tag{3.2}$$

for every measurable set $E \subset \mathbb{R}^n$, since

$$0 = \int_\Omega \mathrm{div} U \, dx = \int_E \mathrm{div} U \, dx + \int_{\Omega \setminus E} \mathrm{div} U \, dx$$

for any U as on the right-hand side of (3.1).

Given $\alpha \in [0, \infty)$, denote

$$\omega_\alpha = \pi^{\alpha/2}/\Gamma(1 + \alpha/2),$$

where

$$\Gamma(s) = \int_0^\infty r^{s-1} e^{-r} \, dr \quad \text{for } s \geq 0,$$

the Euler Gamma function.
Recall that $\Gamma(n + 1) = n!$ if $n \in \mathbb{N}$.
Moreover, ω_n equals the Lebesgue measure of the unit ball in $\mathbb{R}^n$ for $n \in \mathbb{N}$.

Definition 3.2 [Hausdorff measure] Let $\alpha \in [0, \infty)$ and $E \subset \mathbb{R}^n$. For $\varepsilon > 0$, set

$$\mathcal{H}^\alpha_\varepsilon(E) = \inf \left\{ \sum_{k=1}^\infty \frac{\omega_\alpha}{2^\alpha} (\mathrm{diam}(E_k))^\alpha : \mathrm{diam}(E_k) < \varepsilon, E \subset \bigcup_{k=1}^\infty E_k \right\}.$$

The α-dimensional Hausdorff measure of E is defined by

$$\mathcal{H}^\alpha(E) = \lim_{\varepsilon \to 0} \mathcal{H}^\alpha_\varepsilon(E). \tag{3.3}$$

Since the function $\varepsilon \to \mathcal{H}^\alpha_\varepsilon(E)$ is non-increasing, the limit on the right-hand side of (3.3) exists, possibly infinite, for every set $E \subset \mathbb{R}^n$.

Proposition 3.3 [Properties of the Hausdorff measure] *Let $n \in \mathbb{N}$ and $\alpha \in [0, \infty)$.*
(i) $\mathcal{H}^\alpha$ is an outer measure in $\mathbb{R}^n$.
(ii) The Borel sets are $\mathcal{H}^\alpha$-measurable.

Definition 3.4 [Domain of class $C^{m,\alpha}$] Let m be a nonnegative integer, and let $\alpha \in [0, 1]$. An open bounded set $\Omega \in \mathbb{R}^n$ is called a domain of class $C^{m,\alpha}$ if for every $x \in \partial\Omega$ there exist a neighborhood $\mathcal{U}_x$ of x, a Cartesian coordinate system $(\xi_1, \dots \xi_n)$ and a function $\zeta = \zeta(\xi_1, \dots, \xi_{n-1})$ of class $C^{m,\alpha}$ such that

$$\Omega \cap \mathcal{U}_x = \{(\xi_1, \dots \xi_n) \in \mathcal{U}_x : \xi_n > \zeta(\xi_1, \dots, \xi_{n-1})\}. \tag{3.4}$$

We set $C^{m,0} = C^m$.
Domains of class C^∞ are defined accordingly.

Definition 3.5 [Lipschitz domain] A Lipschitz domain is an open set of class $C^{0,1}$.

Definition 3.6 [Domain with the cone property] An open set Ω is said to have the cone property if there exists a finite cone Λ such that each point in Ω is the vertex of a finite cone contained in Ω and congruent to Λ.

Proposition 3.7 *If $E \in C^2$, then*

$$P(E;\Omega) = \mathcal{H}^{n-1}(\partial E \cap \Omega) < \infty. \tag{3.5}$$

Proof. Let $U \in C_0^\infty(\Omega, \mathbb{R}^n)$ be such that $|U(x)| \leq 1$ for $x \in \mathbb{R}^n$. Then, by the Gauss-Green theorem,

$$\int_E \mathrm{div} U(x)\, dx = \int_{\partial E} \mathbf{n} \cdot U d\mathcal{H}^{n-1}(x) \leq \mathcal{H}^{n-1}(\partial E \cap \Omega) < \infty,$$

where $\mathbf{n}(x)$ denotes the outer unit normal to E at x.
In order to prove the reverse inequality, observe that, since $E \in C^2$, there exists a an open set $G \supset \partial E$ such that the distance function $d(x)$ of x from ∂E belongs to $C^1(G \setminus \partial E)$, and

$$\nabla d(x) = \frac{x - \xi(x)}{d(x)}, \tag{3.6}$$

where $\xi(x)$ is the unique point in ∂E such that $d(x) = |x - \xi(x)|$. To verify equation (3.6) observe that $\xi(y)$ is constant as y approaches ∂E along the segment through $\xi(x)$ and parallel to $\mathbf{n}(x)$. Hence, the derivative of d at x along $-\mathbf{n}(x)$ is 1. It is easily seen that d is a Lipschitz function with Lipschitz constant not exceeding 1. Altogether one obtains equation (3.6).
Since $\nabla d(x) = \mathbf{n}(x)$ on ∂E, the latter being a smooth level set $\{d = 0\}$ of the function d, we infer from (3.6) that the function $\mathbf{n}$ has an extension $\widetilde{\mathbf{n}} \in C_0^1(\mathbb{R}^n)$ such that $|\widetilde{\mathbf{n}}(x)| \leq 1$ for $x \in \mathbb{R}^n$. Thus, on choosing $U = \widetilde{\mathbf{n}}\eta$, where $\eta \in C_0^\infty(\Omega)$, $|\eta(x)| \leq 1$, one has that

$$\int_E \mathrm{div} U\, dx = \int_E \mathrm{div}(\widetilde{\mathbf{n}}\eta)\, dx = \int_{\partial E} \eta d\mathcal{H}^{n-1}(x).$$

Hence,

$$\begin{aligned} P(E;\Omega) &\geq \sup\left\{ \int_{\partial E} \eta d\mathcal{H}^{n-1}(x) : \eta \in C_0^\infty(\Omega), |\eta(x)| \leq 1 \right\} \\ &= \mathcal{H}^{n-1}(\partial E \cap \Omega). \end{aligned}$$

□

For an arbitrary measurable set E, one only has that

$$P(E;\Omega) \leq \mathcal{H}^{n-1}(\partial E \cap \Omega). \tag{3.7}$$

Equality can be restored in (3.5) provided that the topological boundary ∂E of E is replaced by a suitable notion of "essential boundary" of E.

In what follows, we denote by $B_r(x)$ the ball in $\mathbb{R}^n$, centered at $x \in \mathbb{R}^n$, having radius $r > 0$. We shall also set $B_r = B_r(0)$.

Definition 3.8 [Essential boundary] Let E be a Lebesgue measurable set in $\mathbb{R}^n$. For $\gamma \in [0,1]$, define

$$E^\gamma = \left\{ x \in \mathbb{R}^n : \lim_{r \to 0^+} \frac{|E \cap B_r(x)|}{|B_r(x)|} = \gamma \right\}.$$

The essential boundary $\partial^M E$ of E is given by

$$\partial^M E = \mathbb{R}^n \setminus (E^0 \cup E^1). \tag{3.8}$$

The set E^γ is called the set of points where E has density γ. Thus, $\partial^M E$ is the set of points of $\mathbb{R}^n$ where E has neither density 0 nor 1.

Theorem 3.9 *Let E be a measurable set in $\mathbb{R}^n$. Then*

$$P(E;\Omega) = \mathcal{H}^{n-1}(\partial^M E \cap \Omega). \tag{3.9}$$

In particular,

$$P(E) = \mathcal{H}^{n-1}(\partial^M E). \tag{3.10}$$

Theorem 3.10 *Let $E \subset \Omega$ such that $|E| < \infty$ and $P(E;\Omega) < \infty$. Then there exists a sequence of sets $\{E_k\}$ such that*
(i) E_k is of class C^∞ in Ω,
(ii) $\lim_{k\to\infty} \chi_{E_k} = \chi_E$ in $L^1_{\mathrm{loc}}(\Omega)$,
(iii) $\lim_{k\to\infty} P(E_k;\Omega) = P(E;\Omega)$.

Given any measurable set $E \subset \mathbb{R}^n$, we denote by $E^\star$ the (open) ball centered at 0 and such that $|E^\star| = |E|$.

Theorem 3.11 [Isoperimetric inequality] *Let E be a measurable set in $\mathbb{R}^n$, $n \geq 1$, with $|E| < \infty$. Then*

$$P(E^\star) \leq P(E). \tag{3.11}$$

Equivalently,

$$n\omega_n^{1/n} |E|^{\frac{1}{n'}} \leq P(E). \tag{3.12}$$

Moreover, equality holds in (3.11) *or* (3.12) *if and only if $E = E^\star$ (up to a set of measure 0).*

4 Symmetrization of functions vanishing on the boundary

In this section, Ω denotes an open subset of $\mathbb{R}^n$.

Definition 4.1 [Radially decreasing symmetral] Let $u \in \mathcal{M}(\Omega)$. The radially decreasing symmetral $u^\bigstar : \Omega^\bigstar \to [0, \infty]$ of u is defined as

$$u^\bigstar(x) = u^*(\omega_n |x|^n) \qquad \text{for } x \in \Omega^\bigstar. \tag{4.1}$$

It is easily seen that

$$\mu_u = \mu_{u^*} = \mu_{u^\bigstar}.$$

Thus, via Proposition 1.6, one has the following.

Proposition 4.2 [Invariance of integrals] *Let $\|\cdot\|_{X(0,|\Omega|)}$ be a rearrangement invariant function norm, and let $u \in \mathcal{M}(\Omega)$. Then*

$$\|u\|_{X(\Omega)} = \|u^\bigstar\|_{X(\Omega^\bigstar)}. \tag{4.2}$$

Besides the invariance of quantities depending only on the distribution function, a fundamental property of the operation of radially decreasing symmetrization, when applied to Sobolev functions, is the non-increase of gradient norms. Such a property is crucial in view of applications of symmetrization to Sobolev type inequalities, and is known as Pólya–Szegö principle. Its proof relies upon the isoperimetric inequality and on the coarea formula.

Given $p \in [1, \infty]$, we define the (homogeneous) Sobolev space

$$V^{1,p}(\Omega) = \{u : \ u \text{ is weakly differentiable in } \Omega \text{ and } |\nabla u| \in L^p(\Omega)\}. \tag{4.3}$$

The local Sobolev space $V^{1,p}_{loc}(\Omega)$ is defined analogously, on replacing $L^p(\Omega)$ by $L^p_{loc}(\Omega)$ on the right-hand side of (4.3).
We also define

$$V^{1,p}_0(\Omega) = \{u \in V^{1,p}(\Omega) : \ \text{the continuation of } u \text{ by } 0 \text{ outside } \Omega \\ \text{is weakly differentiable in } \mathbb{R}^n, \text{ and } \mu_u(t) < \infty \text{ for } t > 0\}.$$

Thus, $V^{1,p}_0(\Omega)$ is the subspace of those functions in $V^{1,p}(\Omega)$ which vanish, in a suitable sense, on $\partial\Omega$.
Parallel definitions hold if $L^p(\Omega)$ is replaced with a more general rearrangement-invariant space $X(\Omega)$. Thus, we set

$$V^1X(\Omega) = \{u : \ u \text{ is weakly differentiable in } \Omega \text{ and } |\nabla u| \in X(\Omega)\}, \tag{4.4}$$

and

$$V^1_0X(\Omega) = \{u \in V^1X(\Omega) : \ \text{the continuation of } u \text{ by } 0 \text{ outside } \Omega \\ \text{is weakly differentiable in } \mathbb{R}^n, \text{ and } \mu_u(t) < \infty \text{ for } t > 0\}.$$

Theorem 4.3 [Coarea formula] *Let* $u \in V^{1,1}_{\rm loc}(\Omega)$ *and let* $f : \Omega \to [0, \infty]$ *be any Borel function. Then, there exists a representative of* u *such that, for a.e.* $t \in \mathbb{R}$,

$$\partial^M\{u > t\} \cap \Omega = \{u = t\} \quad \textit{up to a set of } \mathcal{H}^{n-1}\textit{-measure zero.} \tag{4.5}$$

Moreover,

$$\int_\Omega f(x)|\nabla u|\, dx = \int_{-\infty}^{\infty}\int_{\{u=t\}} f(x)\, d\mathcal{H}^{n-1}(x)dt\,. \tag{4.6}$$

In particular, formula 4.6 tell us that

$$\begin{aligned}\int_\Omega |\nabla u|\, dx &= \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\{u = t\})\, dt\\ &= \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\partial^M\{u > t\} \cap \Omega)\, dt = \int_{-\infty}^{\infty} P(\{u > t\}; \Omega)dt\,.\end{aligned}$$

Theorem 4.4 [Pólya–Szegö principle] *Let* $p \geq 1$, *and let* $u \in V^{1,p}_0(\mathbb{R}^n)$. *Then* $u^\bigstar \in V^{1,p}_0(\mathbb{R}^n)$, *and*

$$\int_{\mathbb{R}^n} |\nabla u^\bigstar|^p\, dx \leq \int_{\mathbb{R}^n} |\nabla u|^p\, dx\,. \tag{4.7}$$

Proof. We split the proof in several steps.
Step 1. *The function* u^* *is locally absolutely continuous in* $(0, \infty)$, *and*

$$\frac{d}{ds}\int_{\{|u|>u^*(s)\}} |\nabla u|\, dx \geq n\omega_n^{\frac{1}{n}} s^{\frac{1}{n'}}(-u^{*\prime}(s)) \quad \text{for a.e. } s > 0.. \tag{4.8}$$

A basic property of Sobolev functions tells us that $|u| \in V^{1,p}_0(\mathbb{R}^n)$ as well, and that $|\nabla|u|| = |\nabla u|$ a.e. in $\mathbb{R}^n$.
Let (a, b) be any subinterval of $(0, \infty)$. We have that $u \in V^{1,1}_{\rm loc}(\mathbb{R}^n)$. By the coarea formula applied with $f = \chi_{\{u^*(b)<|u|<u^*(a)\}}$ and the isoperimetric inequality, one gets that

$$\begin{aligned}\int_{\{u^*(b)<|u|<u^*(a)\}} |\nabla u|\, dx &= \int_{\{u^*(b)<|u|<u^*(a)\}} |\nabla |u||\, dx\\ &= \int_{u^*(b)}^{u^*(a)} \mathcal{H}^{n-1}(\{|u| = t\})\, dt\\ &\geq n\omega_n^{\frac{1}{n}}\int_{u^*(b)}^{u^*(a)} \mu_u(t)^{\frac{1}{n'}}\, dt\\ &\geq n\omega_n^{\frac{1}{n}}\left(\mu_u(u^*(a)^-)\right)^{\frac{1}{n'}}(u^*(a) - u^*(b))\\ &\geq n\omega_n^{\frac{1}{n}} a^{\frac{1}{n'}}(u^*(a) - u^*(b))\,.\end{aligned} \tag{4.9}$$

Note that the last inequality holds owing to property (vi) of Proposition 1.5.

Now, given any $\delta > 0$, let (a_k, b_k), $k = 1, \dots, K$, with $K \in \mathbb{N}$, be any pairwise disjoint intervals contained in (δ, ∞). On applying (4.9) with (a, b) replaced by (a_k, b_k), and adding the resulting inequalities yield

$$\sum_{k=1}^{K} (u^*(a_k) - u^*(b_k)) \leq \frac{1}{n\omega_n^{\frac{1}{n}} \delta^{\frac{1}{n'}}} \int_{\cup_k \{u^*(b_k) < |u| < u^*(a_k)\}} |\nabla u| \, dx \,. \tag{4.10}$$

Since

$$\begin{aligned} | \cup_k \{u^*(b_k) < |u| < u^*(a_k)\}| &= \sum_{k=1}^{K} |\{u^*(b_k) < |u| < u^*(a_k)\}| \\ &= \sum_{k=1}^{K} \big(\mu_u(u^*(b_k)) - \mu_u(u^*(a_k)^-)\big) \\ &\leq \sum_{k=1}^{K} (b_k - a_k) \,, \end{aligned} \tag{4.11}$$

from (4.10) and the Hardy-Littlewood inequality (1.11) we obtain that

$$\sum_{k=1}^{K} (u^*(a_k) - u^*(b_k)) \leq \frac{1}{n\omega_n^{\frac{1}{n}} \delta^{\frac{1}{n'}}} \int_0^{\sum_{k=1}^K (b_k - a_k)} |\nabla u|^*(s) \, ds \,. \tag{4.12}$$

We claim that last integral is convergent. This is trivially verified if $|\nabla u| \in L^\infty(\mathbb{R}^n)$. Otherwise, by the invariance of integrals under decreasing rearrangement,

$$\int_0^\infty |\nabla u|^*(s)^p \, ds = \int_{\mathbb{R}^n} |\nabla u|^p \, dx < \infty \,, \tag{4.13}$$

and hence the assertion follows from the fact that $\lim_{t\to\infty} t^p/t > 0$. The absolute continuity of u^* on (δ, ∞) is thus a consequence of (4.12) and of the absolute continuity of the Lebesgue integral.

In order to prove (4.7), observe that

$$\begin{aligned} n\omega_n^{\frac{1}{n}} \int_{u^*(b_k)}^{u^*(a_k)} \mu_u(t)^{\frac{1}{n'}} \, dt &= n\omega_n^{\frac{1}{n}} \int_{\mu_u(u^*(a_k))}^{\mu_u(u^*(b_k))} \mu_u(u^*(r))^{\frac{1}{n'}} (-u^{*\prime}(r)) \, dr \\ &= n\omega_n^{\frac{1}{n}} \int_{a_k}^{b_k} r^{\frac{1}{n'}} (-u^{*\prime}(r)) \, dr \qquad \text{for } k \in K, \end{aligned} \tag{4.14}$$

where the first equality is a consequence of the (local) absolute continuity of u^*, and the second one holds since $\mu_u(u^*(r)) = r$ if r does not belong to an interval where u^* is constant and $u^{*\prime}$ vanishes in any such interval. From (4.9), (4.12) and the Hardy–Littlewood inequality again, we deduce that, for any family of disjoint intervals $\{(a_k, b_k)\}$ with $(a_k, b_k) \subset (0, \infty)$,

$$\sum_k n\omega_n^{\frac{1}{n}} \int_{a_k}^{b_k} r^{\frac{1}{n'}} (-u^{*\prime}(r)) \, dr \leq \int_0^{\sum_k (b_k - a_k)} |\nabla u|^*(r) \, dr. \tag{4.15}$$

Since each open set in $\mathbb{R}$ is a countable union of disjoint open intervals, inequality (4.15) implies that

$$n\omega_n^{\frac{1}{n}} \int_E r^{\frac{1}{n'}} (-u^{*\prime}(r)) \, dr \leq \int_0^{|E|} |\nabla u|^*(r) \, dr \tag{4.16}$$

for every open set $E \subset (0,1)$. In particular, inequality (4.16) tells us that the function $r^{\frac{1}{n'}}(-u^{*\prime}(r))$ is integrable on $(0,\infty)$. Thanks to the fact that any measurable set can be approximated from outside by open sets, and thanks to the absolute continuity of the Lebesgue integral, inequality (4.16) continues to hold for any measurable set $E \subset (0,\infty)$. Since

$$n\omega_n^{\frac{1}{n}} \int_0^s \Big[(\cdot)^{\frac{1}{n'}}(-u^{*\prime}(\cdot))\Big]^*(r)\,dr = \sup_{|E|=s} n\omega_n^{\frac{1}{n}} \int_E r^{\frac{1}{n'}}(-u^{*\prime}(r))\,dr$$

for $s \in (0,\infty)$, we infer from (4.16) that

$$n\omega_n^{\frac{1}{n}} \int_0^s \Big[(\cdot)^{\frac{1}{n'}}(-u^{*\prime}(\cdot))\Big]^*(r)\,dr \leq \int_0^s |\nabla u|^*(r)\,dr \tag{4.17}$$

for $s \in (0,\infty)$. Hence, by the Hardy–Littlewood–Pólya principle,

$$\|n\omega_n^{\frac{1}{n}} r^{\frac{1}{n'}}(-u^{*\prime}(r))\|_{L^p(0,\infty)} \leq \|\nabla u\|_{L^p(\mathbb{R}^n)}. \tag{4.18}$$

Here, and in similar occurrences below, $\|\nabla u\|_{L^p(\mathbb{R}^n)}$ is an abridged notation for $\|\,|\nabla u|\,\|_{L^p(\mathbb{R}^n)}$. Since u^* is locally absolutely continuous on $(0,\infty)$, then $u^\bigstar$ is weakly differentiable, and

$$|\nabla u^\bigstar(x)| = n\omega_n^{\frac{1}{n}}(\omega_n|x|^n)^{\frac{1}{n'}}(-u^{*\prime}(\omega_n|x|^n)) \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Hence,

$$\|\nabla u^\bigstar\|_{L^p(\mathbb{R}^n)} = \|n\omega_n^{\frac{1}{n}} r^{\frac{1}{n'}}(-u^{*\prime}(r))\|_{L^p(0,\infty)}. \tag{4.19}$$

Inequality (4.7) follows from (4.18) and (4.19). □

The following result is a straightforward consequence of Theorem 4.4

Corollary 4.5 *Let $p \geq 1$ and let Ω be an open subset of $\mathbb{R}^n$. Assume that $u \in V_0^{1,p}(\Omega)$. Then u^* is locally absolutely continuous in $(0,|\Omega|)$, $u^\bigstar \in V_0^{1,p}(\Omega^\bigstar)$ and*

$$\|\nabla u\|_{L^p(\Omega)} \geq \|\nabla u^\bigstar\|_{L^p(\Omega^\bigstar)} = \|n\omega_n^{1/n} s^{1/n'} u^{*\prime}(s)\|_{L^p(0,|\Omega|)}\,. \tag{4.20}$$

Proof. Use Theorem 4.4 and the fact that the continuation of u to $\mathbb{R}^n$ by 0 outside Ω belongs to $V_0^{1,p}(\mathbb{R}^n)$. □

Theorem 4.6 [Pólya–Szegö principle: case of equality] [BZ]; see also [FV] *Let $p > 1$, and let $u \in V_0^{1,p}(\mathbb{R}^n)$. Assume that:*

$$\int_{\mathbb{R}^n} |\nabla u^\bigstar|^p\,dx = \int_{\mathbb{R}^n} |\nabla u|^p\,dx \tag{4.21}$$

and

$$|\{\nabla u^\bigstar = 0\} \cap \{0 < u^\bigstar < \operatorname{esssup} u\}| = 0\,. \tag{4.22}$$

Then

$$u = u^\bigstar \qquad \text{a.e. in } \mathbb{R}^n, \tag{4.23}$$

up to translations.

Via a somewhat different argument in Steps 3 and 4 of the proof of Theorem 4.4 one can easily show that, if equation (4.21) holds, then:

$$u \geq 0;$$

$$\{u > t\} \quad \text{is (equivalent to) a ball for a.e. } t > 0;$$
$$|\nabla u| \quad \text{is constant } \mathcal{H}^{n-1}\text{-a.e on } \{u = t\}, \text{ for a.e. } t > 0.$$

The difficult part in the proof of Theorem 4.6 is to show that, under the additional assumption (4.22), the balls $\{u > t\}$ are concentric.

Owing to inequality (4.17) and to the Hardy–Littlewood–Pólya principle, the following Pólya–Szegö inequality for arbitrary r.i. norms holds.

Theorem 4.7 [Generalized Pólya–Szegö principle] *Let $X(\Omega)$ be an r.i. space, and let $u \in V_0^1X(\Omega)$. Then $u^\bigstar \in V_0^1X(\Omega^\bigstar)$, and*

$$\|\nabla u\|_{X(\Omega)} \geq \|\nabla u^\bigstar\|_{X(\Omega^\bigstar)} = \|n\omega_n^{1/n}s^{1/n'}u^{*\prime}(s)\|_{X(0,|\Omega|)}\,. \tag{4.24}$$

5 Sharp Sobolev inequalities for functions vanishing on the boundary

In this section, Ω denotes an open subset of $\mathbb{R}^n$.

The Pólya–Szegö principle enables one to reduce the problem of any Sobolev type inequality, for functions vanishing on the boundary of their domain, involving arbitrary rearrangement invariant norms, to a suitable one-dimensional inequality for a Hardy type operator.

Theorem 5.1 [Reduction principle for functions vanishing on the boundary] *Let $\|\cdot\|_{X(0,|\Omega|)}$ and $\|\cdot\|_{Y(0,|\Omega|)}$ be rearrangement-invariant function norms. Then*

$$\|u\|_{Y(\Omega)} \leq C\,\|\nabla u\|_{X(\Omega)} \tag{5.1}$$

for some constant C, for every Ω of fixed measure, and every $u \in V_0^1X(\Omega)$ if and only if

$$\left\| n\omega_n^{\frac{1}{n}} \int_t^{|\Omega|} f(s)s^{-\frac{1}{n'}}\,ds \right\|_{Y(0,|\Omega|)} \leq C\,\|f\|_{X(0,|\Omega|)} \tag{5.2}$$

for every nonnegative $f \in X(0,|\Omega|)$.

Proof. Assume that (5.2) holds. Let $u \in V_0^1X(\Omega)$. Since $\|\cdot\|_{Y(0,|\Omega|)}$ is an r.i. function norm, one has that

$$\|u\|_{Y(\Omega)} = \|u^*\|_{Y(0,|\Omega|)}. \tag{5.3}$$

By Theorem 4.7,

$$\|\nabla u\|_{X(\Omega)} \geq \|n\omega_n^{1/n}s^{1/n'}u^{*\prime}(s)\|_{X(0,|\Omega|)}\,. \tag{5.4}$$

On the other hand, since $\lim_{s\to|\Omega|} u^*(s) = 0$,

$$u^*(s) = \int_s^{|\Omega|} (-u^{*\prime}(r))\,dr \quad \text{for } s \in (0,|\Omega|). \tag{5.5}$$

Inequality (5.1) follows from (5.3)-(5.5), via (5.2) applied with $f(s) = n\omega_n^{1/n}s^{1/n'}u^{*\prime}(s)$.

Conversely, (5.2) follows from (5.1) on choosing a ball as domain Ω in (5.1), and considering radially decreasing trial functions u in (5.1). Indeed, equality holds in (5.4) for any such u. □

Theorem 5.1 can be exploited to exhibit the optimal constants in the Sobolev inequalities in $V_0^{1,p}(\Omega)$.

Lemma 5.2 *Let $0 < p \le q$. Then*

$$\int_0^\infty \varphi(t)^{\frac{q}{p}} d(t^q) \le \left(\int_0^\infty \varphi(t) d(t^p) \right)^{\frac{q}{p}}, \tag{5.6}$$

for every non-increasing function $\varphi : (0,\infty) \to [0,\infty)$.

Proof. The change of variable $t = \tau^{\frac{1}{p}}$, $\alpha = \frac{q}{p}$, on both sides of (5.6) shows that it suffices to show that

$$\int_0^\infty \varphi(t)^\alpha d(t^\alpha) \le \left(\int_0^\infty \varphi(t) dt \right)^\alpha, \tag{5.7}$$

for $\alpha \ge 1$ and for every non-increasing function $\varphi : (0,\infty) \to [0,\infty)$. Inequality (5.7) follows from the chain

$$\begin{aligned} \int_0^\infty \varphi(t)^\alpha d(t^\alpha) &= \alpha \int_0^\infty (t\varphi(t))^{\alpha-1} \varphi(t) dt \\ &\le \alpha \int_0^\infty \left(\int_0^t \varphi(\tau) d\tau \right)^{\alpha-1} \varphi(t) dt \\ &= \left(\int_0^\infty \varphi(t) dt \right)^\alpha . \end{aligned} \tag{5.8}$$

□

Theorem 5.3 [Sharp Sobolev inequality in $V_0^{1,1}(\mathbb{R}^n)$] [Ma1, FF] *The inequality*

$$n\omega_n^{\frac{1}{n}} \|u\|_{L^{n'}(\mathbb{R}^n)} \le \|\nabla u\|_{L^1(\mathbb{R}^n)} \tag{5.9}$$

holds for every $u \in V_0^{1,1}(\mathbb{R}^n)$. The constant $n\omega_n^{\frac{1}{n}}$ is sharp, although it is not attained.

Proof. If $u \in V_0^{1,1}(\mathbb{R}^n)$, then $|u| \in V_0^{1,1}(\mathbb{R}^n)$, and $\|\nabla u\|_{L^1(\mathbb{R}^n)} = \|\nabla |u|\|_{L^1(\mathbb{R}^n)}$. Thus, we may assume, without loss of generality, that $u \ge 0$. By the isoperimetric inequality in $\mathbb{R}^n$,

$$n\omega_n^{\frac{1}{n}} \mu_u(t)^{\frac{1}{n'}} = n\omega_n^{\frac{1}{n}} |\{u > t\}|^{\frac{1}{n'}} \le P(\{u > t\}) \qquad \text{for a.e. } t \ge 0. \tag{5.10}$$

Thus, by the coarea formula, inequality (5.10), Lemma 5.2, and equation (1.9)

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla u| dx = \int_0^\infty P(\{u > t\}) dt &\ge n\omega_n^{\frac{1}{n}} \int_0^\infty \mu_u(t)^{\frac{1}{n'}} dt \\ &\ge n\omega_n^{\frac{1}{n}} \left(\int_0^\infty \mu_u(t)\, d(t^{n'}) \right)^{\frac{1}{n'}} = n\omega_n^{\frac{1}{n}} \|u\|_{L^{n'}(\mathbb{R}^n)}. \end{aligned} \tag{5.11}$$

Inequality (5.9) follows. Next, given any $\varepsilon > 0$, consider the Lipschitz continuous function $u_\varepsilon : \mathbb{R}^n \to [0,1]$ given by

$$u_\varepsilon(x) = \begin{cases} 1 & \text{if } |x| \le 1, \\ 1 - \frac{|x|-1}{\varepsilon} & \text{if } 1 < |x| < 1+\epsilon, \\ 0 & \text{if } |x| \ge 1+\epsilon. \end{cases} \tag{5.12}$$

Since $|\nabla u_\varepsilon(x)| = 1/\varepsilon$ if $1 < |x| < 1+\epsilon$, and vanishes a.e. elsewhere, it is easily verified that

$$\lim_{\varepsilon \to 0+} \frac{\|\nabla u_\varepsilon\|_{L^1(\mathbb{R}^n)}}{\|u_\varepsilon\|_{L^{n'}(\mathbb{R}^n)}} = n\omega_n^{\frac{1}{n}}.$$

This shows the sharpness of the constant in (5.9). □

Theorem 5.4 [Sharp Sobolev inequality in $V_0^{1,p}(\mathbb{R}^n)$, $1 < p < n$] [Ta1, Au] *Let $1 < p < n$. Then*

$$C(p,n)\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq \|\nabla u\|_{L^p(\mathbb{R}^n)} \tag{5.13}$$

for every $u \in V_0^{1,p}(\mathbb{R}^n)$, where

$$C(p,n) = \pi^{\frac{1}{2}} n^{\frac{1}{p}} \left(\frac{n-p}{p-1}\right)^{\frac{1}{p'}} \left(\frac{\Gamma(n/p)\Gamma(1+n-n/p)}{\Gamma(n)\Gamma(1+n/2)}\right)^{\frac{1}{n}}. \tag{5.14}$$

Equality holds in (5.13) *if*

$$u(x) = \frac{a}{(b + |x-x_0|^{p'})^{\frac{n}{p}-1}} \quad \textit{for } x \in \mathbb{R}^n, \tag{5.15}$$

for some $a \in \mathbb{R}$, $b > 0$ and $x_0 \in \mathbb{R}^n$.

Proof . By the reduction principle (Theorem 5.1), it suffices to prove the statement for radially decreasing functions. The result for these functions follows is equivalent to the Bliss inequality contained in the next theorem. □

Theorem 5.5 [Bliss inequality] *Let $n \geq 2$, $1 < p < n$, and let $C(p,n)$ be the constant given by* (5.14). *Then*

$$C(p,n)\left(n\omega_n \int_0^\infty f(r)^{p^*} r^{n-1}\,dr\right)^{1/p^*} \leq \left(n\omega_n \int_0^\infty (-f'(r))^p r^{n-1}\,dr\right)^{1/p} \tag{5.16}$$

for every decreasing, locally absolutely continuous function $f : [0,\infty) \to [0,\infty)$. Equality holds in (5.16) *if*

$$f(r) = \frac{a}{(b + r^{p'})^{\frac{n}{p}-1}} \tag{5.17}$$

for some $a, b > 0$.

Proof Approximation, scaling and normalization arguments allow us to assume that f is a continuously differentiable function such that

$$\int_0^\infty |f'(r)|^p r^{n-1}\,dr < \infty$$

and

$$\int_0^\infty f(r)^{p^*} r^{n-1}\,dr = 1\,. \tag{5.18}$$

Let $g : (0,\infty) \to [0,\infty)$ be a continuous function, such that

$$\int_0^\infty g(r)^{p^*} r^{n-1}\,dr = \int_0^\infty g(r)^{p^*} r^{p'+n-1}\,dr = 1. \tag{5.19}$$

Define $T : [0,\infty) \to [0,\infty)$ by

$$\int_0^s f(r)^{p^*} r^{n-1}\,dr = \int_0^{T(s)} g(\varrho)^{p^*} \varrho^{n-1}\,d\varrho\,. \tag{5.20}$$

Thus,

$$g(T(s)) = T(s)^{-\frac{n-1}{p^*}} T'(s)^{-\frac{1}{p^*}} f(s) s^{\frac{n-1}{p^*}}, \tag{5.21}$$

whence, by Young inequality,

$$\begin{aligned} g(T(s))^{p^*-\frac{p^*}{n}} T(s)^{n-1} T'(s) &= f(s)^{p^*-\frac{p^*}{n}} s^{n-1} \left[\left(\frac{T(s)}{s} \right)^{n-1} T'(s) \right]^{\frac{1}{n}} \\ &\leq \frac{1}{n} f(s)^{p^*-\frac{p^*}{n}} s^{n-1} \left[(n-1) \frac{T(s)}{s} + T'(s) \right] \\ &\frac{1}{n} f(s)^{p^*-\frac{p^*}{n}} (s^{n-1} T(s))' \quad \text{for } s \in [0, \infty). \end{aligned} \tag{5.22}$$

Note that equality holds in (5.22) if and only if $T(s) = cs$ for some $c > 0$. Integration by parts and Hölder inequality yield

$$\begin{aligned} &\int_0^\infty f(s)^{p^*-\frac{p^*}{n}} (s^{n-1} T(s))' \, ds \\ &= -\left(p^* - \frac{p^*}{n} \right) \int_0^\infty f(s)^{p^*-\frac{p^*}{n}-1} f'(s) s^{n-1} T(s) \, ds \\ &\leq \left(p^* - \frac{p^*}{n} \right) \int_0^\infty f(s)^{p^*-\frac{p^*}{n}-1} |f'(s)| s^{n-1} T(s) \, ds \\ &\leq \left(p^* - \frac{p^*}{n} \right) \left(\int_0^\infty f(s)^{p^*} s^{n-1} T(s)^{p'} \, ds \right)^{\frac{1}{p'}} \left(\int_0^\infty |f'(s)|^p s^{n-1} \, ds \right)^{\frac{1}{p}} \\ &= \left(p^* - \frac{p^*}{n} \right) \left(\int_0^R g(\varrho)^{p^*} \varrho^{p'+n-1} \, d\varrho \right)^{\frac{1}{p'}} \left(\int_0^\infty |f'(s)|^p s^{n-1} \, ds \right)^{\frac{1}{p}} \\ &= \left(p^* - \frac{p^*}{n} \right) \left(\int_0^\infty |f'(s)|^p s^{n-1} \, ds \right)^{\frac{1}{p}}. \end{aligned} \tag{5.23}$$

Combining (5.22) and (5.23) yields

$$\frac{n(n-p)}{p(n-1)} \int_0^\infty g(\varrho)^{p^*-\frac{p^*}{n}} \varrho^{n-1} \, d\varrho \leq \left(\int_0^\infty |f'(s)|^p s^{n-1} \, ds \right)^{\frac{1}{p}}. \tag{5.24}$$

Inequality (5.24) continues to hold for any nonnegative continuous function g. Moreover, an inspection of the proof shows that equality holds in (5.24) if

$$T(s) = cs$$

for some and

$$f'(s) = -k f(s)^{\frac{n}{n-p}} T(s)^{\frac{1}{p-1}} = -k c^{\frac{1}{p-1}} f(s)^{\frac{n}{n-p}} s^{\frac{1}{p-1}}$$

for some positive constants c and k. Hence, we infer that equality holds in (5.24) if f is as in (5.17), and hence g has the same form, with a and b such that (5.19) hold. This choice of g in (5.24) yields (5.16). □

Theorem 5.6 [Sharp Sobolev inequality in $V_0^{1,p}(\mathbb{R}^n)$, $p > n$] [Ta2] *Let $p > n$. Then*

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq n^{-1/p} \omega_n^{-1/n} \left(\frac{p-1}{p-n} \right)^{1/p'} |\mathrm{supp}\,(u)|^{\frac{1}{n}-\frac{1}{p}} \|\nabla u\|_{L^p(\mathbb{R}^n)} \tag{5.25}$$

for every $u \in V_0^{1,p}(\mathbb{R}^n)$ *with* $|\operatorname{supp}(u)| < \infty$. *Equality holds in* (5.25) *if*

$$u(x) = \begin{cases} a\big(b^{\frac{p-n}{p-1}} - |x - x_0|^{\frac{p-n}{p-1}}\big) & \text{if } |x - x_0| \leq b \\ 0 & \text{otherwise}, \end{cases} \tag{5.26}$$

for some $a \in \mathbb{R}$, $b \geq 0$ *and* $x_0 \in \mathbb{R}^n$.

Proof. We have that

$$\|u\|_{L^\infty(\mathbb{R}^n)} = u^*(0) = \int_0^{|\operatorname{supp}(u)|} (-u^*)'(s)\, ds\,.$$

Owing to Hölder inequality,

$$\begin{aligned} &\int_0^{|\operatorname{supp}(u)|} (-u^*)'(s)\, ds \\ &\leq \bigg(\int_0^{|\operatorname{supp}(u)|} s^{-\frac{p'}{n'}}\, ds\bigg)^{\frac{1}{p'}} \bigg(\int_0^{|\operatorname{supp}(u)|} (s^{\frac{1}{n'}}(-u^*)'(s))^p\, ds\bigg)^{\frac{1}{p}} \\ &= n^{\frac{1}{p'}} \Big(\frac{p-1}{p-n}\Big)^{1/p'} |\operatorname{supp}(u)|^{\frac{1}{n}-\frac{1}{p}} \bigg(\int_0^{|\operatorname{supp}(u)|} (s^{\frac{1}{n'}}(-u^*)'(s))^p\, ds\bigg)^{\frac{1}{p}}. \end{aligned}$$

By the Pólya–Szegö principle (4.7),

$$\int_{\mathbb{R}^n} |\nabla u|^p\, dx \geq \int_{\mathbb{R}^n} |\nabla u^\bigstar|^p\, dx = \int_0^{|\operatorname{supp}(u)|} (n\omega_n^{\frac{1}{n}} s^{\frac{1}{n'}} (-u^*)'(s))^p\, ds\,.$$

Altogether, inequality (5.25) follows.
The argument above shows that equality holds in (5.25) provided that u is radially decreasing, and fulfils

$$(-u^*)'(s) = \begin{cases} cs^{-\frac{p'}{n'}} & \text{if } s \in (0, |\operatorname{supp}(u)|) \\ 0 & \text{if } s \geq |\operatorname{supp}(u)| \end{cases}$$

for some positive constant c. This is the case when u has the form (5.26). □

Theorem 5.7 *Inequalities* (5.9), (5.13) *and* (5.25) *hold even if* $\mathbb{R}^n$ *is replaced with any open subset* Ω. *The constants in* (5.9) *and* (5.13) *are still sharp in the resulting inequalities, but, in contrast with* (5.13), *the constant in the corresponding inequality is never attained if* $\Omega \neq \mathbb{R}^n$.

In the borderline case when $p = n$ (and $|\Omega| < \infty$), one can show that, if $q < \infty$, then

$$\|u\|_{L^q(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)} \tag{5.27}$$

for some constant $C = C(|\Omega|, n, p, q)$, and for every $u \in V_0^{1,p}(\Omega)$. On the other hand, inequality (5.27) fails if $q = \infty$.
However, a stronger inequality than (5.27) does hold.

Theorem 5.8 [Moser inequality] [Mo] *Let* $n \geq 2$. *Then there exists a constant* $C = C(n, |\operatorname{sprt} u|)$ *such that*

$$\frac{1}{|\operatorname{sprt} u|} \int_{\mathbb{R}^n} e^{\left(\frac{n\omega_n^{1/n}|u(x)|}{\|\nabla u\|_{L^n(\mathbb{R}^n)}}\right)^{n'}} dx \leq C \tag{5.28}$$

for every $u \in V_0^{1,n}(\mathbb{R}^n)$ *with* $0 < |\operatorname{sprt} u| < \infty$. *The constant* $n\omega_n^{1/n}$ *is best possible, in that inequality (5.28) fails for for any real number* C *if* $n\omega_n^{1/n}$ *is replaced by a larger number.*

The proof of Theorem 5.8 again makes use of the the Pólya–Szegö principle of Corollary 4.5, and of the one-dimensional inequality contained in the next lemma.

Lemma 5.9 *Let* $p \in (1, \infty)$, *and let*

$$m_p = \sup_\phi \int_0^\infty e^{\phi(s)^{p'} - s} ds \,, \tag{5.29}$$

where ϕ *ranges among all non-decreasing, locally absolutely continuous functions in* $[0, \infty)$ *fulfilling* $\phi(0) = 0$ *and* $\int_0^\infty \phi'(s)^p ds \leq 1$. *Then*

$$m_p < \infty \,. \tag{5.30}$$

Incidentally, observe that the quantity m_p given by (5.29) remains unchanged if the class of trial functions is enlarged to include also not necessarily (positive and) monotone functions ϕ, provided that $\phi(s)^{p'}$ is replaced with $|\phi(s)|^{p'}$ and $\phi'(s)^p$ with $|\phi'(s)|^p$. This can be easily seen on replacing $\phi(s)$ with $\int_0^s |\phi'(s)| ds$.

Proof of Lemma 5.9. For each $t \in \mathbb{R}$, define

$$E_t = \{s \geq 0 : s - \phi(s)^{p'} \leq t\},$$

and let

$$c_p = \frac{7}{1 - (1 + 2^{1-p})^{\frac{1}{1-p}}}.$$

From the fact that $\phi(0) = 0$, the Hölder inequality and the fact that $\int_0^\infty \phi'(s)^p ds \leq 1$, we have that

$$\phi(s)^{p'} = \left(\int_0^s \phi'(r) dr\right)^{p'} \leq s \qquad \text{for } s > 0.$$

Hence,

$$E_t = \emptyset \,, \qquad \text{for } t < 0. \tag{5.31}$$

We now show that

$$|E_t| \leq (c_p + 2)t \,, \qquad \text{for } t > 0. \tag{5.32}$$

Inequality (5.32) trivially holds if $E_t \subset [0, 2t]$. If this is not the case, then inequality (5.32) will follow if we prove that

$$s_2 - s_1 \leq c_p t \tag{5.33}$$

for every $s_1, s_2 \in E_t$ satisfying $2t \leq s_1 < s_2$. From the definition of E_t and the Hölder inequality,

$$s_1 - t \leq \left(\int_0^{s_1} \phi'(r) dr\right)^{p'} \leq s_1 \left(\int_0^{s_1} \phi'(r)^p dr\right)^{\frac{1}{p-1}} \leq s_1 \left(1 - \int_{s_1}^\infty \phi'(r)^p dr\right)^{\frac{1}{p-1}}.$$

Thus,

$$\int_{s_1}^\infty \phi'(r)^p dr \leq 1 - \left(1 - \frac{t}{s_1}\right)^{p-1}. \tag{5.34}$$

From the definition of E_t, the Hölder inequality again, and (5.34),

$$\begin{aligned} s_2 - t &\leq \Big(\int_0^{s_1} \phi'(r)dr + \int_{s_1}^{s_2} \phi'(r)dr \Big)^{p'} \\ &\leq \left[s_1^{\frac{1}{p'}} \Big(\int_0^{s_1} \phi'(r)^p dr \Big)^{\frac{1}{p}} + (s_2 - s_1)^{\frac{1}{p'}} \Big(\int_{s_1}^{s_2} \phi'(r)^p dr \Big)^{\frac{1}{p}} \right]^{p'} \\ &\leq \left[s_1^{\frac{1}{p'}} + (s_2 - s_1)^{\frac{1}{p'}} \Big(\int_{s_1}^{\infty} \phi'(r)^p dr \Big)^{\frac{1}{p}} \right]^{p'} \\ &\leq \left[s_1^{\frac{1}{p'}} + (s_2 - s_1)^{\frac{1}{p'}} \Big(1 - \Big(1 - \frac{t}{s_1} \Big)^{p-1} \Big)^{\frac{1}{p}} \right]^{p'} . \end{aligned}$$

Therefore,

$$\frac{s_2}{s_1} - \frac{t}{s_1} \leq \left[1 + \Big(\frac{s_2}{s_1} - 1 \Big)^{\frac{1}{p'}} \Big(1 - \Big(1 - \frac{t}{s_1} \Big)^{p-1} \Big)^{\frac{1}{p}} \right]^{p'} . \tag{5.35}$$

Set $M = \frac{s_2 - s_1}{t}$ and $z = 1 - \frac{t}{s_1}$. Then, $\frac{1}{2} \leq z < 1$ and (5.35) can be rewritten as

$$M(1-z) + z \leq \big(1 + M^{\frac{1}{p'}} (1-z)^{\frac{1}{p'}} (1 - z^{p-1})^{\frac{1}{p}} \big)^{p'} . \tag{5.36}$$

The convexity of the function $\tau \mapsto \tau^{p'}$ entails that, for $\lambda \in [0,1]$,

$$\big(1 + M^{\frac{1}{p'}} (1-z)^{\frac{1}{p'}} (1 - z^{p-1})^{\frac{1}{p}} \big)^{p'} \leq (1-\lambda)^{1-p'} + \lambda^{1-p'} M (1-z)(1 - z^{p-1})^{\frac{1}{p-1}} . \tag{5.37}$$

Choosing $\lambda = 1 - z^{2(p-1)}$ and combining (5.36) and (5.37) yield

$$M \leq \frac{(1-\lambda)^{1-p'} - z}{(1-z)\big(1 - \lambda^{1-p'}(1 - z^{p-1})^{\frac{1}{p-1}}\big)} = \frac{z^{-2} + z^{-1} + 1}{1 - (1 + z^{p-1})^{\frac{1}{1-p}}} \leq c_p \, ,$$

whence (5.33) follows.

The Layer Cake principle and a simple change of variables imply that

$$\int_0^\infty e^{-g(s)}\, ds = \int_{-\infty}^\infty |\{ s > 0 : g(s) \leq t \}| \, e^{-t}\, dt$$

for every measurable function $g : (0, \infty) \to \mathbb{R}$. Thus,

$$\int_0^\infty e^{\phi(s)^{p'} - s} ds = \int_{-\infty}^\infty |E_t| e^{-t} dt \leq c_p + 2 \, ,$$

where the inequality is a consequence of (5.31) and (5.32). □

Proof of Theorem 5.8. By (1.7), we have

$$\int_{\mathbb{R}^n} e^{(n\omega_n^{1/n} |u(x)|)^{n'}} dx = \int_0^{|\mathrm{sprt}\, u|} e^{(n\omega_n^{1/n} u^*(s))^{n'}} ds \, . \tag{5.38}$$

On the other hand, the Pólya-Szegö principle (4.7) tells us that

$$\|\nabla u\|_{L^n(\mathbb{R}^n)} \geq \left(\int_0^{|\mathrm{sprt}\, u|} \left(n\omega_n^{1/n} s^{1/n'} (-{u^*}'(s))\right)^n ds \right)^{1/n} . \tag{5.39}$$

Owing to (5.38) and (5.39), inequality (5.28) will follow if we show that, for each $a > 0$,

$$\sup_\psi \frac{1}{a} \int_0^a e^{(n\omega_n^{1/n}\psi(s))^{n'}} ds = m_n \,, \tag{5.40}$$

as ψ ranges among all non-increasing locally absolutely continuous function $\psi : (0, a] \to [0, \infty)$ such that $\psi(a) = 0$ and

$$\int_0^a \left(n\omega_n^{1/n} s^{1/n'} (-\psi'(s))\right)^n ds \leq 1.$$

Given such a ψ, define the non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ by

$$\phi(t) = n\omega_n^{1/n} \psi(ae^{-t}), \qquad \text{for } t > 0. \tag{5.41}$$

Note that $\phi(0) = 0$. The change of variable

$$s = ae^{-t} \tag{5.42}$$

gives

$$\int_0^\infty \phi'(t)^n \, dt = \int_0^a \left(n\omega_n^{1/n} s^{1/n'} (-\psi'(s))\right)^n ds \;\leq\; 1 \,,$$

and

$$\int_0^\infty e^{\phi(t)^{n'}} e^{-t} dt = \frac{1}{a} \int_0^a e^{(n\omega_n^{1/n}\psi(s))^{n'}} ds \,.$$

Hence, equation (5.40) follows from Lemma 5.9, since, for each fixed a, the class of functions appearing in definition (5.29) agrees with the class of functions ϕ given by (5.41) with ψ as above.

The sharpness of the constant $n\omega_n^{1/n}$ in (5.28) can be verified on testing the inequality on the sequence $\{u_k\}_{k \in \mathbb{N}}$ of radially decreasing functions defined as

$$u_k(x) = \begin{cases} \frac{k^{1/n'}}{n\omega_n^{1/n}} & \text{if } \; |x| \leq e^{-k/n} \\ \frac{k^{-1/n}}{\omega_n^{1/n}} \log\left(\frac{1}{|x|}\right) & \text{if } \; e^{-k/n} < |x| \leq 1 \\ 0 & \text{otherwise} \,. \end{cases}$$

□

6 Relative isoperimetric inequalities

In this Section, Ω denotes an open set in $\mathbb{R}^n$, $n \geq 2$, having finite measure.

Definition 6.1 [Isoperimetric function] [Ma1, Ma2] The isoperimetric function $I_\Omega : [0, |\Omega|] \to [0, \infty)$ is defined as

$$I_\Omega(s) = \inf\{P(E; \Omega) : E \subset \Omega, s \leq |E| \leq |\Omega|/2\} \qquad \text{for } s \in [0, |\Omega|/2], \tag{6.1}$$

and $I_\Omega(s) = I_\Omega(|\Omega| - s)$ if $s \in (|\Omega|/2, |\Omega|]$.

The very definition of I_Ω leads to the $r$elative isoperimetric inequality in Ω

$$I_\Omega(|E|) \le P(E;\Omega) \tag{6.2}$$

for every $E \subset \Omega$.

The function I_Ω is explicitly known only for special sets Ω (e.g. when Ω is a ball [Ci1]).

The next result ensures that, whenever Ω is connected, inequality (6.2) contains nontrivial information, in that its left-hand side is strictly positive if $0 < |E| < |\Omega|/2$.

Theorem 6.2 *Assume that Ω is connected. Then*

$$I_\Omega(s) > 0 \qquad \textit{for } s \in (0,|\Omega|). \tag{6.3}$$

In view of applications, quantitative information is needed on the isoperimetric function I_Ω. A basic result in this connection deals with the case when Ω is a Lipschitz domain.

Theorem 6.3 *Let Ω be a connected Lipschitz domain in $\mathbb{R}^n$. Then there exists a constant C such that*

$$C\min\{s, |\Omega| - s\}^{\frac{1}{n'}} \le I_\Omega(s) \qquad \textit{for } s \in [0,|\Omega|]. \tag{6.4}$$

Combining (6.2) and (6.4) yields the following relative isoperimetric inequality on any connected Lipschitz domain $\Omega \subset \mathbb{R}^n$:

$$C\min\{|E|, |\Omega| - |E|\}^{\frac{1}{n'}} \le P(E;\Omega) \tag{6.5}$$

for some constant C and for every $E \subset \Omega$.

For Sobolev functions which do not vanish on the boundary of their domain, no n-dimensional symmmetrization principle in the form of Theorem 4.4 is available. Nevertheless, an inequality involving the signed one-dimensional rearrangement of u and the isoperimetric function of Ω holds, provided that Ω is connected.

Theorem 6.4 [CEG] *Let $p \ge 1$. Assume that Ω is connected, and that $u \in V^{1,p}(\Omega)$. Then u° is locally absolutely continuous in $(0,|\Omega|)$, and*

$$\|\nabla u\|_{L^p(\Omega)} \ge \|I_\Omega(s) u^{\circ\prime}(s)\|_{L^p(0,|\Omega|)}.$$

Corollary 6.5 *Let $p \ge 1$. Assume that Ω is a connected Lipschitz domain, and that $u \in V^{1,p}(\Omega)$. Then u° is locally absolutely continuous in $(0,|\Omega|)$, and there exists a constant $C = C(\Omega)$ such that*

$$\|\nabla u\|_{L^p(\Omega)} \ge C\| \min\{s, |\Omega| - s\}^{1/n'} u^{\circ\prime}(s)\|_{L^p(0,|\Omega|)}. \tag{6.6}$$

An analogue of Theorem 6.4 in arbitrary rearrangement invariant spaces holds.

Theorem 6.6 *Assume that Ω is connected. Let $X(\Omega)$ be a rearrangement-invariant space, and let $u \in V^1X(\Omega)$. Then u° is locally absolutely continuous in $(0,|\Omega|)$, and*

$$\|\nabla u\|_{X(\Omega)} \ge \|I_\Omega(s) u^{\circ\prime}(s)\|_{X(0,|\Omega|)}. \tag{6.7}$$

7 Sobolev inequalities for functions which need not vanish on the boundary

In this Section, Ω denotes a connected open set in $\mathbb{R}^n$, $n \geq 2$, having finite measure.

Let $X(\Omega)$ be an r.i. space. Recall that, according to our definition, a function $u \in V^1X(\Omega)$ if $|\nabla u| \in X(\Omega)$. This assumption does not entail, in general, that u belongs to $X(\Omega)$ as well, and not even to $L^1(\Omega)$. Examples of domains for which $V^{1,2}(\Omega) \not\subseteq L^1(\Omega)$ are, for instance, those of Nykodým type [Ma2, Sections 5.2 and 5.4].

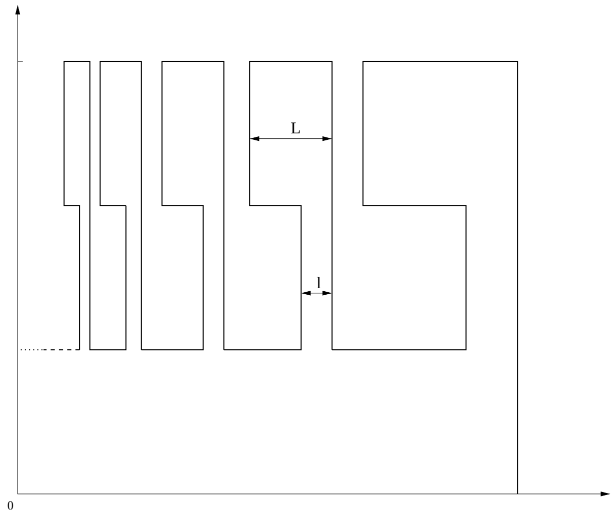

Figure 1: Nikodým example

However, if $I_\Omega(s)$ does not decay at 0 faster than linearly, namely if there exists a positive constant C such that

$$I_\Omega(s) \geq Cs \quad \text{for } s \in [0, \tfrac{|\Omega|}{2}], \tag{7.1}$$

then any function $u \in V^1X(\Omega)$ does at least belong to $L^1(\Omega)$, for any rearrangement invariant space $X(\Omega)$.

Proposition 7.1 [Condition for $V^1L^1(\Omega) \subset L^1(\Omega)$] *Assume that* (7.1) *holds. Then* $V^1L^1(\Omega) \subset L^1(\Omega)$, *and*

$$\frac{C}{2}\left\| u - \frac{1}{|\Omega|}\int_\Omega u\,dx \right\|_{L^1(\Omega)} \leq \|\nabla u\|_{L^1(\Omega)} \tag{7.2}$$

for every $u \in V^1L^1(\Omega)$, *where* C *is the same constant as in* (7.1).

Proof. Let $\operatorname{med}(u)$ denote the median of a function $u \in \mathcal{M}(\Omega)$, given by

$$\operatorname{med}(u) = \sup\{t \in \mathbb{R} : \ |\{x \in \Omega : \ u(x) > t\}| > \tfrac{|\Omega|}{2}\} \ (= u^\circ(\tfrac{|\Omega|}{2})).$$

We claim that

$$C\|u - \operatorname{med}(u)\|_{L^1(\Omega)} \leq \|\nabla u\|_{L^1(\Omega)} \tag{7.3}$$

for every $u \in V^1L^1(\Omega)$. On replacing, if necessary, u by $u - \operatorname{med}(u)$, we may assume, without loss of generality, that $\operatorname{med}(u) = 0$. Thus,

$$|\{u_\pm > t\}| \leq \tfrac{|\Omega|}{2} \qquad \text{for } t > 0. \tag{7.4}$$

By (6.2) and (7.1),

$$P(\{u_\pm > t\}, \Omega) \geq I_\Omega(|\{u_\pm > t\}|) \geq C|\{u_\pm > t\}|.$$

Therefore, owing to (7.4), and to the coarea formula, we have that

$$\begin{aligned} C\|u_\pm\|_{L^1(\Omega)} &= C\int_0^\infty |\{u_\pm > t\}|\, dt \leq \int_0^\infty P(\{u_\pm > t\}, \Omega)\, dt \\ &= \int_0^\infty \int_{\partial^M\{u_\pm > t\}\cap\Omega} d\mathcal{H}^{n-1}(x)\, dt = \int_\Omega |\nabla u_\pm| dx. \end{aligned} \tag{7.5}$$

Hence, (7.3) follows. In particular, (7.3) tells us that $V^1L^1(\Omega) \subset L^1(\Omega)$. Inequality (7.2) is a consequence of (7.3) and of the fact that

$$\left\| u - \frac{1}{|\Omega|}\int_\Omega u\, dx \right\|_{L^1(\Omega)} \leq 2\|u - \mathrm{med}(u)\|_{L^1(\Omega)}$$

for every $u \in L^1(\Omega)$. □

Under (7.1), an assumption which will always be kept in force hereafter, $V^1X(\Omega)$ is a Banach space, equipped with the norm

$$\|u\|_{V^1X(\Omega)} = \|u\|_{L^1(\Omega)} + \|\nabla u\|_{X(\Omega)}.$$

We also define the subspace $V^1_\perp X(\Omega)$ of $V^1X(\Omega)$ as

$$V^1_\perp X(\Omega) = \left\{ u \in V^1X(\Omega) : \int_\Omega u\, dx = 0, \right\}.$$

The gradient rearrangement inequality contained in Theorem 6.6 enables one to reduce the problem of Sobolev embeddings of $V^1X(\Omega)$ into $Y(\Omega)$, for any r.i. spaces $X(\Omega)$ and $Y(\Omega)$, and of corresponding Sobolev-Poincaré type inequalities, to one-dimensional inequalities for a Hardy type operator.

This reduction principle depends only on a lower bound for the isoperimetric function I_Ω of Ω in terms of some other non-decreasing function $I : [0, 1] \to [0, \infty)$; precisely, on the existence of a positive constant c such that

$$I_\Omega(s) \geq cI(cs) \quad \text{for } s \in [0, \tfrac{|\Omega|}{2}]. \tag{7.6}$$

In the light of assumption (7.1), we shall require that

$$\inf_{t\in(0,|\Omega|)} \frac{I(t)}{t} > 0. \tag{7.7}$$

Theorem 7.2 [Reduction principle in $V^1X(\Omega)$] [CP1] *Assume that Ω fulfils (7.6) for some function I satisfying (7.7). Let $\|\cdot\|_{X(0,|\Omega|)}$ and $\|\cdot\|_{Y(0,|\Omega|)}$ be rearrangement-invariant function norms. If there exists a constant C_1 such that*

$$\left\| \int_t^{|\Omega|} \frac{f(s)}{I(s)}\, ds \right\|_{Y(0,|\Omega|)} \leq C_1 \|f\|_{X(0,|\Omega|)} \tag{7.8}$$

for every nonnegative $f \in X(0, |\Omega|)$, then

$$V^1X(\Omega) \to Y(\Omega), \tag{7.9}$$

and there exists a constant C_2 such that

$$\|u\|_{Y(\Omega)} \leq C_2 \|\nabla u\|_{X(\Omega)} \tag{7.10}$$

for every $u \in V^1_\perp X(\Omega)$.

The Sobolev embedding (7.9) (or the Poincaré inequality (7.10)) and inequality (7.8) (with $I \approx I_\Omega$) are actually equivalent in customary families of domains Ω. Loosely speaking, this is the case whenever the geometry of Ω allows for the construction of a family of trial functions u in (7.9) or (7.10) characterized by the following properties:
(i) the level sets of u are isoperimetric (or almost isoperimetric) in Ω;
(ii) $|\nabla u|$ is constant (or almost constant) on the boundary of the level sets of u.

A basic case when this situation occurs is when Ω is a Lipschitz domain, or, more generally, a John domain.
Recall that a bounded open set Ω in $\mathbb{R}^n$ is called a *John domain* if there exist a constant $c \in (0,1)$ and a point $x_0 \in \Omega$ such that for every $x \in \Omega$ there exists a rectifiable curve $\varpi : [0,l] \to \Omega$, parameterized by arclength, such that $\varpi(0) = x$, $\varpi(l) = x_0$, and

$$\mathrm{dist}\,(\varpi(r), \partial\Omega) \geq cr \qquad \text{for } r \in [0,l].$$

Then we have what follows.

Theorem 7.3 [Reduction principle for John domains] *Assume that Ω is a John domain in $\mathbb{R}^n$. Let $\|\cdot\|_{X(0,|\Omega|)}$ and $\|\cdot\|_{Y(0,|\Omega|)}$ be rearrangement-invariant function norms. Then the following assertions are equivalent.*
(i) *The Hardy type inequality*

$$\left\| \int_t^{|\Omega|} f(s) s^{-\frac{1}{n'}}\, ds \right\|_{Y(0,|\Omega|)} \leq C_1 \|f\|_{X(0,|\Omega|)} \tag{7.11}$$

holds for some constant C_1, and for every nonnegative $f \in X(0,|\Omega|)$.
(ii) *The Sobolev embedding*

$$V^1 X(\Omega) \to Y(\Omega) \tag{7.12}$$

holds.
(iii) *The Poincaré inequality*

$$\|u\|_{Y(\Omega)} \leq C_2 \|\nabla u\|_{X(\Omega)} \tag{7.13}$$

holds for some constant C_2 and every $u \in V^1_\perp X(\Omega)$.

Proof, sketched. One can show that

$$I_\Omega(s) \approx s^{\frac{1}{n'}} \quad \text{for } s \text{ near } 0,$$

for any John domain. Thus, by Theorem 7.2, the Hardy inequality (7.11) implies the Sobolev emebdding (7.12) and the Poincaré inequality (7.13). The converse implications follow on choosing radially decreasing (compactly supported) functions in Ω. □

Given a rearrangement-invariant function norm $\|\cdot\|_{X(0,|\Omega|)}$, we define $\|\cdot\|_{X_{\mathrm{John}}(0,|\Omega|)}$ as the rearrangement-invariant function norm whose associate function norm is given by

$$\|f\|_{X'_{\mathrm{John}}(0,|\Omega|)} = \left\| s^{-\frac{1}{n'}} \int_0^s f^*(r)dr \right\|_{X'(0,|\Omega|)} \tag{7.14}$$

for $f \in \mathcal{M}_+(0,|\Omega|)$. The function norm $\|\cdot\|_{X_{\mathrm{John}(0,|\Omega|)}}$ is optimal, as a target, for Sobolev embeddings of $V^1X(\Omega)$.

Theorem 7.4 [Optimal target for John domains] *Let Ω and $\|\cdot\|_{X(0,|\Omega|)}$ be as in Theorem 7.3. Then the functional $\|\cdot\|_{X'_{\mathrm{John}}(0,|\Omega|)}$, given by* (7.14)*, is a rearrangement-invariant function norm, whose associate norm $\|\cdot\|_{X_{\mathrm{John}}(0,|\Omega|)}$ satisfies*

$$V^1X(\Omega) \to X_{\mathrm{John}}(\Omega), \tag{7.15}$$

and

$$\|u\|_{X_{\mathrm{John}}(\Omega)} \le C\, \|\nabla u\|_{X(\Omega)} \tag{7.16}$$

for some constant C and every $u \in V^1_\perp X(\Omega)$.
Moreover, the function norm $\|\cdot\|_{X_{\mathrm{John}}(0,|\Omega|)}$ is optimal in (7.15) *and* (7.16) *among all rearrangement-invariant norms.*

Proof. Owing to Theorem 7.3, inequality (7.16) holds if (and only if)

$$\left\| \int_t^{|\Omega|} f(s) s^{-\frac{1}{n'}}\, ds \right\|_{X_{\mathrm{John}}(0,|\Omega|)} \le C\, \|f\|_{X(0,|\Omega|)} \tag{7.17}$$

for some constant C, and for every nonnegative $f \in X(0,|\Omega|)$. By the very definition of the associate norm and by Fubini's theorem, we have that

$$\begin{aligned}
&\sup_{\|f\|_{X(0,|\Omega|)}\le 1} \left\| \int_t^{|\Omega|} f(s) s^{-\frac{1}{n'}}\, ds \right\|_{X_{\mathrm{John}}(0,|\Omega|)} \\
&= \sup_{\|f\|_{X(0,|\Omega|)}\le 1} \ \sup_{\|g\|_{X'_{\mathrm{John}}(0,|\Omega|)}\le 1} \int_0^{|\Omega|} g^*(t) \int_t^{|\Omega|} f(s) s^{-\frac{1}{n'}}\, ds\, dt \\
&= \sup_{\|g\|_{X'_{\mathrm{John}}(0,|\Omega|)}\le 1} \ \sup_{\|f\|_{X(0,|\Omega|)}\le 1} \int_0^{|\Omega|} f(s) s^{-\frac{1}{n'}} \int_0^s g^*(t)dt\, ds \\
&= \sup_{\|g\|_{X'_{\mathrm{John}}(0,|\Omega|)}\le 1} \left\| s^{-\frac{1}{n'}} \int_0^s g^*(t)dt \right\|_{X'(0,|\Omega|)} \\
&= \sup_{\|g\|_{X'_{\mathrm{John}}(0,|\Omega|)}\le 1} \|g\|_{X'_{\mathrm{John}}(0,|\Omega|)} = 1.
\end{aligned} \tag{7.18}$$

Hence, (7.17) follows.

It remains to show that the function norm $\|\cdot\|_{X_{\mathrm{John}}(0,|\Omega|)}$ is optimal in (7.17). To this purpose, suppose that$\|\cdot\|_{Y(0,|\Omega|)}$ is another r.i. function norm such that (7.15), or (7.16), holds. By Theorem 7.3 again, this is equivalent to

$$\left\| \int_t^{|\Omega|} f(s) s^{-\frac{1}{n'}}\, ds \right\|_{Y(0,|\Omega|)} \le C\, \|f\|_{X(0,|\Omega|)} \tag{7.19}$$

for some constant C, and for every nonnegative $f \in X(0,|\Omega|)$. Via a chain analogous to (7.18), one can deduce from this inequality that

$$\left\| s^{-\frac{1}{n'}} \int_0^s g^*(t)dt \right\|_{X'(0,|\Omega|)} \le C \, \|g\|_{Y'(0,|\Omega|)}$$

for every $g \in Y'(0,|\Omega|)$. The last inequality is equivalent to the embedding $Y'(0,|\Omega|) \to X'_{\mathrm{John}}(0,|\Omega|)$, which is in turn equivalent to $X_{\mathrm{John}}(0,|\Omega|) \to Y(0,|\Omega|)$. This shows the optimality of $X_{\mathrm{John}}(0,|\Omega|)$. □

We conclude this Section by presenting a few Sobolev embeddings on a John domain Ω (in particular, a Lipschitz domain), which can be established via Theorem 7.3 or Theorem 7.4.

Classical Sobolev embeddings [So, Ga, Ni, Po, Tr, Yu]. Let $1 \le p \le \infty$. Then

$$V^1L^p(\Omega) \to \begin{cases} L^{\frac{np}{n-p}}(\Omega) & \text{if } 1 \le p < n, \\ \exp L^{n'}(\Omega) & \text{if } p = n, \\ L^\infty(\Omega) & \text{if } p > n. \end{cases} \tag{7.20}$$

Embeddings for Lorentz-Sobolev spaces [On, Pe, BW, Han]. Assume that either $p \in (1,\infty]$ and $q \in [1,\infty]$, or $p = 1$ and $q = 1$. Then

$$V^1L^{p,q}(\Omega) \to \begin{cases} L^{\frac{np}{n-p},q}(\Omega) & \text{if } 1 \le p < n \text{ and } q \in [1,\infty], \\ L^{\infty,q;-1}(\Omega) & \text{if } p = n \text{ and } q > 1, \\ L^\infty(\Omega) & \text{if either } p = n \text{ and } q = 1, \text{ or } .p > n. \end{cases} \tag{7.21}$$

Here, $L^{\infty,q;-1}(\Omega)$ denotes the Lorentz-Zygmund space associated with the r.i. function norm given by

$$\|f\|_{L^{\infty,q}(\log L)^{-1}(0,|\Omega|)} = \|s^{-\frac{1}{q}} \, (1+\log \tfrac{1}{s})^{-1} f^*(s)\|_{L^q(0,|\Omega|)} \tag{7.22}$$

for $\in \mathcal{M}_+(0,|\Omega|)$.
The target spaces in (7.21) are optimal among all r.i. spaces.

Embeddings for Orlicz-Sobolev spaces [Ci2, Ci3]. Let A be a Young function. We may assume, without loss of generality, that

$$\int_0 \left(\frac{t}{A(t)}\right)^{\frac{1}{n-1}} dt < \infty. \tag{7.23}$$

Indeed, by (2.9), the function A can be modified near 0, if necessary, in such a way that (7.23) is fulfilled, and the space $V^1L^A(\Omega)$ is unchanged (up to equivalent norms).

If the integral

$$\int^\infty \left(\frac{t}{A(t)}\right)^{\frac{1}{n-1}} dt \tag{7.24}$$

diverges, we define the function $H : [0,\infty) \to [0,\infty)$ as

$$H(s) = \left(\int_0^s \left(\frac{t}{A(t)}\right)^{\frac{1}{n-1}} dt\right)^{\frac{1}{n'}} \qquad \text{for } s \ge 0, \tag{7.25}$$

and the Young function A_n as

$$A_n(t) = A(H^{-1}(t)) \qquad \text{for } t \geq 0. \tag{7.26}$$

Then

$$V^1L^A(\Omega) \to \begin{cases} L^{A_n}(\Omega) & \text{if the integral (7.24) diverges,} \\ L^\infty(\Omega) & \text{if the integral (7.24) converges.} \end{cases} \tag{7.27}$$

Moreover, the target spaces in (7.27) are optimal among all Orlicz spaces.

The optimal targets for $V^1L^A(\Omega)$ among all rearrangement-invariant spaces can also be characterized.

8 Higher-order Sobolev embeddings

Let Ω be an open subset of $\mathbb{R}^n$, and let $m \in \mathbb{N}$. We define the m-th order Sobolev space $V^mX(\Omega)$ as

$$V^mX(\Omega) = \big\{u : u \text{ is } m\text{-times weakly differentiable in } \Omega, \text{ and } |\nabla^m u| \in X(\Omega)\big\}. \tag{8.1}$$

Here, $\nabla^m u$ denotes the vector of all m-th order weak derivatives of u. We shall also denote $\nabla^0 u = u$. Let us notice that in the definition of $V^mX(\Omega)$ it is only required that the derivatives of the highest order m of u belong to $X(\Omega)$. As already observed in the case when $m = 1$, this assumption does not entail, in general, that also u and its derivatives up to the order $m-1$ belong to $X(\Omega)$, and even to $L^1(\Omega)$.
Analogously to the case when $m = 1$, if Ω has finite measure and satisfies

$$I_\Omega(s) \geq Cs \quad \text{for } s \in [0, \tfrac{|\Omega|}{2}], \tag{8.2}$$

then $V^mX(\Omega)$ is a Banach space, equipped with the norm

$$\|u\|_{V^mX(\Omega)} = \sum_{k=0}^{m-1} \|\nabla^k u\|_{L^1(\Omega)} + \|\nabla^m u\|_{X(\Omega)}. \tag{8.3}$$

Under (8.2), we also define the subspace $V^m_\perp X(\Omega)$ of $V^mX(\Omega)$ as

$$V^m_\perp X(\Omega) = \left\{u \in V^mX(\Omega) : \int_\Omega \nabla^k u \, dx = 0, \ \text{for } \ k = 0, \dots, m-1\right\}. \tag{8.4}$$

A reduction principle for higher-order Sobolev embeddings and Poincaré type inequalities holds in the spirit of Theorem 7.2.
As in the first-order case, this reduction principle depends only on (7.6), namely on the existence of a non-decreasing function $I : [0, 1] \to [0, \infty)$ and of a positive constant c such that

$$I_\Omega(s) \geq cI(cs) \quad \text{for } s \in [0, \tfrac{|\Omega|}{2}]. \tag{8.5}$$

In view of assumption (8.2), we assume that I fulfils (8.2), i.e.

$$\inf_{t\in(0,|\Omega|)} \frac{I(t)}{t} > 0. \tag{8.6}$$

In the remaining part of this Section, we assume that Ω is connected and has finite measure.

Theorem 8.1 [Higher-order reduction principle] [CPS] *Assume that Ω fulfils* (8.5) *for some non-decreasing function I satisfying* (8.6). *Let $m \in \mathbb{N}$, and let $\|\cdot\|_{X(0,|\Omega|)}$ and $\|\cdot\|_{Y(0,|\Omega|)}$ be rearrangement-invariant function norms. If there exists a constant C_1 such that*

$$\left\| \int_t^1 \frac{f(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{m-1} ds \right\|_{Y(0,|\Omega|)} \leq C_1 \|f\|_{X(0,|\Omega|)} \tag{8.7}$$

for every nonnegative $f \in X(0,|\Omega|)$, then

$$V^m X(\Omega) \to Y(\Omega), \tag{8.8}$$

and there exists a constant C_2 such that

$$\|u\|_{Y(\Omega)} \leq C_2 \|\nabla^m u\|_{X(\Omega)} \tag{8.9}$$

for every $u \in V_\perp^m X(\Omega)$.

Remark 8.2 Under the assumptions of Theorem 8.1, the Sobolev embedding (8.8) and the Poincaré inequality (8.9) can be shown to be equivalent. On the other hand, as in the first-order case, properties (8.7) and (8.8) (or (8.9)) need not be equivalent on an arbitrary (typically very irregular) domain. However, heuristically speaking, properties (8.7), (8.8) and (8.9) turn out to be equivalent for $m > 1$ on the same domains Ω as for $m = 1$. Such equivalence certainly holds in any customary, non-pathological situation, such as, for instance, on John domains, and, in particular, on Lipschitz domains.

Remark 8.3 A reduction theorem in the spirit of Theorem 8.1, concerning the compactness of embeddings of the form (8.8), is established in [Sl].

Now we are in a position to characterize the optimal rearrangement-invariant target space in the Sobolev embedding (8.8), at least in the situation discussed in Remark 8.2. Such an optimal space is the one associated with the rearrangement-invariant function norm $\|\cdot\|_{X_{m,I}(0,|\Omega|)}$, whose associate norm is defined as

$$\|f\|_{X'_{m,I}(0,|\Omega|)} = \left\| \frac{1}{I(s)} \int_0^s \left(\int_t^s \frac{dr}{I(r)} \right)^{m-1} f^*(t)dt \right\|_{X'(0,|\Omega|)} \tag{8.10}$$

for $f \in \mathcal{M}_+(0,|\Omega|)$.

Theorem 8.4 [Optimal higher-order target] *Assume that Ω, m, I and $\|\cdot\|_{X(0,|\Omega|)}$ are as in Theorem 8.1. Then the functional $\|\cdot\|_{X'_{m,I}(0,|\Omega|)}$, given by* (8.10), *is a rearrangement-invariant function norm, whose associate norm $\|\cdot\|_{X_{m,I}(0,|\Omega|)}$ satisfies*

$$V^m X(\Omega) \to X_{m,I}(\Omega), \tag{8.11}$$

and there exists a constant C such that

$$\|u\|_{X_{m,I}(\Omega)} \leq C \|\nabla^m u\|_{X(\Omega)} \tag{8.12}$$

for every $u \in V_\perp^m X(\Omega)$.
Moreover, if Ω is such that (8.8) *(or equivalently* (8.9)*) implies* (8.7), *and hence* (8.7), (8.8) *and* (8.9) *are equivalent, then the function norm $\|\cdot\|_{X_{m,I}(0,|\Omega|)}$ is optimal in* (8.11) *and* (8.12) *among all rearrangement-invariant norms.*

An important special case of Theorems 8.1 and 8.4 is enucleated in the following corollary.

Corollary 8.5 [Sobolev embeddings into L^∞] *Assume that Ω, m, I and $\|\cdot\|_{X(0,|\Omega|)}$ are as in Theorem 8.1. If*

$$\left\| \frac{1}{I(s)} \left(\int_0^s \frac{dr}{I(r)} \right)^{m-1} \right\|_{X'(0,|\Omega|)} < \infty \,, \tag{8.13}$$

then

$$V^m X(\Omega) \to L^\infty(\Omega), \tag{8.14}$$

and there exists a constant C such that

$$\|u\|_{L^\infty(\Omega)} \le C \, \|\nabla^m u\|_{X(\Omega)} \tag{8.15}$$

for every $u \in V^m_\perp X(\Omega)$.
Moreover, if Ω is such that (8.8) *(or equivalently* (8.9)*) implies* (8.7)*, and hence* (8.7), (8.8) *and* (8.9) *are equivalent, then* (8.13) *is necessary for* (8.14) *or* (8.15) *to hold.*

Remark 8.6 If Ω is such that (8.8) (eqiv. (8.9)) implies (8.7), and hence (8.7), (8.8) and (8.9) are equivalent, then (8.14) cannot hold, whatever $\|\cdot\|_{X(0,|\Omega|)}$ is, if I decays so fast at 0 that

$$\int_0 \frac{dr}{I(r)} = \infty.$$

Our last main result concerns the preservation of optimality in targets among all rearrangement-invariant spaces under iteration of Sobolev embeddings of arbitrary order.

Theorem 8.7 [Iteration principle] [CPS] *Assume that Ω, I and $\|\cdot\|_{X(0,|\Omega|)}$ are as in Theorem 8.1. Let $k, h \in \mathbb{N}$. Then*

$$(X_{k,I})_{h,I}(\Omega) = X_{k+h,I}(\Omega), \tag{8.16}$$

up to equivalent norms.

We now specialize the reduction principle to a few instances.
We begin with the reduction theorem for John domains.

Theorem 8.8 [Reduction principle for John domains] [CPS] *Let $n \in \mathbb{N}$, $n \ge 2$, and let $m \in \mathbb{N}$. Assume that Ω is a John domain in $\mathbb{R}^n$. Let $\|\cdot\|_{X(0,|\Omega|)}$ and $\|\cdot\|_{Y(0,|\Omega|)}$ be rearrangement-invariant function norms. Then the following assertions are equivalent.*
(i) *The Hardy type inequality*

$$\left\| \int_t^1 f(s) s^{-1+\frac{m}{n}} \, ds \right\|_{Y(0,|\Omega|)} \le C_1 \, \|f\|_{X(0,|\Omega|)} \tag{8.17}$$

holds for some constant C_1, and for every nonnegative $f \in X(0,|\Omega|)$.
(ii) *The Sobolev embedding*

$$V^m X(\Omega) \to Y(\Omega) \tag{8.18}$$

holds.
(iii) *The Poincaré inequality*

$$\|u\|_{Y(\Omega)} \le C_2 \, \|\nabla^m u\|_{X(\Omega)} \tag{8.19}$$

holds for some constant C_2 and every $u \in V^m_\perp X(\Omega)$.

Versions of the above results also hold in a domain Ω in $\mathbb{R}^n$, $n \geq 1$, equipped with a finite measure ν, which is absolutely continuous with respect to the Lebesgue measure and enjoys some mild properties. A classical instance is that of the Gauss space, corresponding to the case when $\Omega = \mathbb{R}^n$ endowed with the probability measure

$$d\gamma_n(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} dx.$$

The Gauss measure of a set $E \subset \mathbb{R}^n$ is thus given by

$$\gamma_n(E) = (2\pi)^{-\frac{n}{2}} \int_E e^{-\frac{|x|^2}{2}} dx\,,$$

and the Gaussian perimeter by

$$P_{\gamma_n}(E) = (2\pi)^{-\frac{n}{2}} \int_{\partial^M E} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x).$$

The isoperimetric function $I_{\mathbb{R}^n,\gamma_n} : [0,1] \to [0,\infty]$ in Gauss space is accordingly defined as

$$I_{(\mathbb{R}^n,\gamma_n)}(s) = \inf\left\{P_{\gamma_n}(E) : E \subset \mathbb{R}^n, s \leq \gamma_n(E) \leq \tfrac{1}{2}\right\} \quad \text{if } s \in [0, \tfrac{1}{2}], \tag{8.20}$$

and $I_{(\mathbb{R}^n,\gamma_n)}(s) = I_{(\mathbb{R}^n,\gamma_n)}(1-s)$ if $s \in (\frac{1}{2}, 1]$. The isoperimetric inequality in $(\mathbb{R}^n, \gamma_n)$ then reads

$$P_{\gamma_n}(E) \geq I_{(\mathbb{R}^n,\gamma_n)}(\gamma_n(E)), \tag{8.21}$$

where E is any measurable subset of $\mathbb{R}^n$. The isoperimetric theorem in Gauss space tells us that equality holds in (8.21) if (and only if) E is equivalent to a half-space in $\mathbb{R}^n$. In other words, half-spaces minimize Gaussian perimeter among all subsets of $\mathbb{R}^n$ of prescribed Gauss measure. In particular, one has that

$$I_{(\mathbb{R}^n,\gamma_n)}(s) \approx s\sqrt{\log \tfrac{2}{s}} \quad \text{for } s \in (0, \tfrac{1}{2}]. \tag{8.22}$$

Given $m \in \mathbb{N}$ and a rearrangement-invariant space $X(\mathbb{R}^n, \gamma_n)$, we define the m-th order Gaussian Sobolev space $V^m X(\mathbb{R}^n, \gamma_n)$ as

$$V^m X(\mathbb{R}^n, \gamma_n) \;=\; \left\{u \;:\; u \text{ is } m\text{-times weakly differentiable in } \mathbb{R}^n, \text{ and } |\nabla^m u| \in X(\mathbb{R}^n, \gamma_n)\right\}.$$

One has that $V^m X(\mathbb{R}^n, \gamma_n) \subset L^1(\mathbb{R}^n, \gamma_n)$ for every rearrangement-invariant space $X(\mathbb{R}^n, \gamma_n)$, and hence the subspace

$$V^m_\perp X(\mathbb{R}^n, \gamma_n) = \left\{u \in V^m X(\mathbb{R}^n, \gamma_n) : \int_{\mathbb{R}^n} \nabla^k u \, d\gamma_n = 0, \text{ for } k = 0, \dots, m-1\right\}$$

is well defined.
However, one may have

$$V^m X(\mathbb{R}^n, \gamma_n) \not\subseteq X(\mathbb{R}^n, \gamma_n)$$

in general. This is the case, for example, when $X(\mathbb{R}^n, \gamma_n) = L^\infty(\mathbb{R}^n, \gamma_n)$, or when $X(\mathbb{R}^n, \gamma_n) = \exp L^\beta(\mathbb{R}^n, \gamma_n)$ for some $\beta > 0$.

The first-order reduction principle in Gauss space reads as follows.

Theorem 8.9 [First-order reduction principle in Gauss space] [CP1] *Let $n \in \mathbb{N}$, and let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Then the following facts are equivalent.*
(i) *The inequality*

$$\left\| \int_s^1 \frac{f(r)}{r\sqrt{\log \frac{2}{r}}}\, dr \right\|_{Y(0,1)} \leq C_1 \, \|f\|_{X(0,1)} \tag{8.23}$$

holds for some constant C_1, and for every nonnegative $f \in X(0,1)$.
(ii) *The embedding*

$$V^1 X(\mathbb{R}^n, \gamma_n) \to Y(\mathbb{R}^n, \gamma_n) \tag{8.24}$$

holds.
(iii) *The Poincaré inequality*

$$\|u\|_{Y(\mathbb{R}^n,\gamma_n)} \leq C_2 \, \|\nabla u\|_{X(\mathbb{R}^n,\gamma_n)} \tag{8.25}$$

holds for some constant C_2, and for every $u \in V^1_\perp X(\mathbb{R}^n, \gamma_n)$.

A standard Gaussian Sobolev embedding in $V^{1,p}(\mathbb{R}^n, \gamma_n)$ tell us that, if $1 \leq p < \infty$, then

$$V^{1,p}(\mathbb{R}^n, \gamma_n) \to L^p(\log L)^{\frac{p}{2}}(\mathbb{R}^n, \gamma_n). \tag{8.26}$$

In the limiting case when $p = \infty$, one has that

$$V^{1,\infty}(\mathbb{R}^n, \gamma_n) \to \exp L^2(\mathbb{R}^n, \gamma_n). \tag{8.27}$$

Note that there is a loss in the degree of integrability between first-order derivatives of a function and the function itself in the last embedding. Such a loss also appears in embeddings for exponential type Gaussian Sobolev spaces. Indeed, if $0 < \beta < \infty$, then

$$V^1 \exp L^\beta(\mathbb{R}^n, \gamma_n) \to \exp L^{\frac{2\beta}{2+\beta}}(\mathbb{R}^n, \gamma_n). \tag{8.28}$$

Embeddings (8.26)–(8.28) can be established via the reduction principle contained in Theorem 8.9, which also ensures the optimality of the targets spaces among all r.i. spaces.

Specialization of (a suitably generalized version of) Theorem 8.1 to the case of Gauss space easily leads to the following reduction principle for Gaussian Sobolev embeddings of any order.

Theorem 8.10 [Higher-order Reduction principle in Gauss space] [CPS] *Let $n \in \mathbb{N}$, $m \in \mathbb{N}$, and let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Then the following facts are equivalent.*
(i) *The inequality*

$$\left\| \frac{1}{\left(\log \frac{2}{s}\right)^{\frac{m}{2}}} \int_s^1 \frac{f(r)}{r} \left(\log \frac{r}{s}\right)^{m-1} dr \right\|_{Y(0,1)} \leq C_1 \, \|f\|_{X(0,1)}$$

holds for some constant C_1, and for every nonnegative $f \in X(0,1)$.
(ii) *The embedding*

$$V^m X(\mathbb{R}^n, \gamma_n) \to Y(\mathbb{R}^n, \gamma_n)$$

holds.
(iii) *The Poincaré inequality*

$$\|u\|_{Y(\mathbb{R}^n,\gamma_n)} \leq C_2 \, \|\nabla^m u\|_{X(\mathbb{R}^n,\gamma_n)}$$

holds for some constant C_2, and for every $u \in V^m_\perp X(\mathbb{R}^n, \gamma_n)$.

A characterization of the optimal target r.i. spaces in arbitrary-order Gaussian Sobolev spaces can be provided. Moreover, a sharp iteration principle, in the spirit of Theorem 8.7, for the optimal r.i. target spaces holds in Gaussian Sobolev embeddings.

9 Sobolev trace embeddings

In the present section we assume that Ω is a bounded domain with the cone condition in $\mathbb{R}^n$, $n \geq 2$. Under this assumption, one can show that

$$V^m X(\Omega) = W^m X(\Omega), \tag{9.1}$$

for every rearrangement-invariant space $X(\Omega)$, where

$$\begin{aligned} W^m X(\Omega) = \{u \in X(\Omega) :\ & u \text{ is } m \text{ times weakly differentiable in } \Omega \\ & \text{and } |\nabla^k u| \in X(\Omega) \text{ for } k = 0, \dots, m\}\,. \end{aligned}$$

This follows as a special case of [CPS, Proposition 4.5].
When $X(\Omega) = L^p(\Omega)$, we denote $W^m L^p(\Omega)$ simply by $W^{m,p}(\Omega)$, as usual.
Given any $d \in \mathbb{N}$ such that $1 \leq d \leq n$, we call Ω_d the (non empty) intersection of Ω with a d-dimensional affine subspace of $\mathbb{R}^n$.
If

$$d \geq n - m, \tag{9.2}$$

then a linear trace operator

$$\mathrm{Tr} : W^{m,1}(\Omega) \to L^1(\Omega_d) \tag{9.3}$$

is classically well defined at any function in $W^{m,1}(\Omega)$ via approximation by smooth functions. Here, $L^1(\Omega_d)$ stands for a Lebesgue space on Ω_d with respect to the d-dimensional Hausdorff measure $\mathcal{H}^d$. When $d = n$, one has that $\Omega_n = \Omega$, and Tr is the identity operator.

In what follows we present a reduction theorem for Sobolev trace embeddings of the form

$$\mathrm{Tr}\, u : W^m X(\Omega) \to Y(\Omega_d), \tag{9.4}$$

and for corresponding Poincaré trace inequalities. Here, $X(\Omega)$ and $Y(\Omega_d)$ are rearrangement-invariant spaces, and d, m, n are subject to (9.2).
An ensuing characterization of the optimal target $Y(\Omega_d)$ in (9.4), and a related sharp iteration principle will also be stated.
Let us preliminarily comment on assumption (9.2). Since $X(\Omega) \to L^1(\Omega)$ for any rearrangement-invariant space, provided that Ω has finite measure, one has that $W^m X(\Omega) \to W^{m,1}(\Omega)$ for any $m \in \mathbb{N}$ and any such space $X(\Omega)$. Thus, by (9.3), under assumption (9.2) the trace operator Tr is certainly well defined from $W^m X(\Omega)$ into $L^1(\Omega_d)$ (at least), whatever $m \in \mathbb{N}$ and $X(\Omega)$ are. On the other hand, dropping this assumption (in the case when $m < n$) would exclude Sobolev type spaces built upon rearrangement-invariant spaces $X(\Omega)$ endowed with a too weak norm, for instance $L^1(\Omega)$. Since we are not going to impose any restriction on the rearrangement-invariant space $X(\Omega)$, condition (9.2) has to be kept in force throughout.
The reduction principle for trace embeddings reads as follows.

Theorem 9.1 [Reduction principle for trace embeddings] [CP2] *Let Ω be a bounded open set with the cone property in $\mathbb{R}^n$, $n \geq 2$. Assume that $m \in \mathbb{N}$ and $d \in \mathbb{N}$ are such that $1 \leq d \leq n$ and $d \geq n-m$, and let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Then the following facts are equivalent.*
(i) *The Sobolev trace embedding*

$$\mathrm{Tr} : W^m X(\Omega) \to Y(\Omega_d) \tag{9.5}$$

holds.
(ii) *The Poincaré trace inequality*

$$\|\mathrm{Tr} u\|_{Y(\Omega_d)} \leq C_1 \|\nabla^m u\|_{X(\Omega)} \tag{9.6}$$

holds for some constant C_1, and for every $u \in W^m_\perp X(\Omega)$.
(iii) *The inequality*

$$\left\| \int_{t^{\frac{n}{d}}}^1 f(s)\, s^{-1+\frac{m}{n}}\, ds \right\|_{Y(0,1)} \leq C_2 \|f\|_{X(0,1)} \tag{9.7}$$

holds for some constant C_2 and for every nonnegative $f \in X(0,1)$.

Remark 9.2 The statement of Theorem 9.1 is uninteresting in the case when $m \geq n$, since then assertions (i)–(iii) hold for any rearrangement-invariant norms $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$.

Theorem 9.1 is the key step in a description of the optimal target space in (9.4).
Given $n, m, d \in \mathbb{N}$ such that $1 \leq d \leq n$ and $d \geq n-m$, we call $\|\cdot\|_{X^m_{d,n}(0,1)}$ the rearrangement-invariant function norm whose associate function norm is given by

$$\|f\|_{(X^m_{d,n})'(0,1)} = \left\| s^{-1+\frac{m}{n}} \int_0^{s^{\frac{d}{n}}} f^*(r)dr \right\|_{X'(0,1)} \tag{9.8}$$

for every $f \in \mathcal{M}_+(0,1)$.

Theorem 9.3 [Optimal target spaces for trace embeddings] [CP2] *Let Ω be a bounded open set with the cone property in $\mathbb{R}^n$, $n \geq 2$. Assume that $m \in \mathbb{N}$ and $d \in \mathbb{N}$ are such that $1 \leq d \leq n$ and $d \geq n-m$, and let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Let $\|\cdot\|_{X^m_{d,n}(0,1)}$ be the rearrangement-invariant function norm obeying* (9.8). *Then*

$$\mathrm{Tr} : W^m X(\Omega) \to X^m_{d,n}(\Omega_d). \tag{9.9}$$

Moreover, the space $X^m_{d,n}(\Omega_d)$ is optimal in (9.9) *among all rearrangement-invariant spaces.*

An important special case of Theorem 9.3 is enucleated in the following corollary, which provides us with a characterization of the Sobolev spaces $W^m X(\Omega)$ which are mapped into $L^\infty(\Omega_d)$ by the trace operator.

Corollary 9.4 [Trace embeddings into L^∞] *Let n, d, m, Ω, and $\|\cdot\|_{X(0,1)}$ be as in Theorem 9.3. Then the following facts are equivalent:*

$$\mathrm{Tr} : W^m X(\Omega) \to L^\infty(\Omega_d); \tag{9.10}$$

$$X^m_{d,n}(\Omega_d) = L^\infty(\Omega_d); \tag{9.11}$$

$$\|s^{-1+\frac{m}{n}}\|_{X'(0,1)} < \infty. \tag{9.12}$$

In particular, (9.10) *and* (9.11) *hold for any rearrangement invariant function norm $\|\cdot\|_{X(0,1)}$, provided that $m \geq n$.*

A proof of Theorem 9.1 makes use of a two-step iteration method, which relies on the characterization of the optimal rearrangement-invariant target space in (arbitrary-order) Sobolev embeddings in the whole of Ω described in Section 8, and on a characterization of the optimal rearrangement-invariant target space in trace embeddings in a special case, namely on hyperplanes. The latter is just a special case of Theorem 9.3 corresponding to $d = n-1$, and follows from the special case of Theorem 9.1, which amounts to the following statement.

Theorem 9.5 *Let Ω be a bounded open set with the cone property in $\mathbb{R}^n$, $n \geq 2$, and let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Then the Sobolev trace embedding*

$$\mathrm{Tr} : W^1X(\Omega) \to Y(\Omega_{n-1}) \tag{9.13}$$

holds if and only if the Hardy type inequality

$$\left\| \int_{t^{\frac{n}{n-1}}}^1 f(s)\, s^{-1+\frac{1}{n}}\, ds \right\|_{Y(0,1)} \leq C\, \|f\|_{X(0,1)} \tag{9.14}$$

holds for some constant C and for every nonnegative function $f \in X(0,1)$.

A proof of Theorem 9.5 is based on an interpolation argument which makes use of Peetre's K-functional.

Remark 9.6 A version of Theorem 9.5 holds with Ω_{n-1} replaced with $\partial\Omega$, and the trace operator on Ω_{n-1} replaced with the trace operator on $\partial\Omega$ – see [CKP, Theorem 3.1].

A remarkable feature of the approach to Theorem 9.1 outlined above is that any composition of Sobolev and/or trace embedding, involving an optimal rearrangement-invariant target space, results in a Sobolev trace embedding whose target is still optimal among all rearrangement-invariant spaces. This sharp iteration principle is the subject of the next theorem.

Theorem 9.7 [Sharp iteration principle for trace embeddings] [CP2] *Let Ω be an open set with the cone property in $\mathbb{R}^n$, $n \geq 2$. Let $k, h, d, \ell \in \mathbb{N}$ be such that $1 \leq d \leq \ell \leq n$, $\ell \geq n-k$ and $d \geq \ell - h$. Assume that $\Omega_d \subset \Omega_\ell$. Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Then*

$$(X^k_{\ell,n})^h_{d,\ell}(\Omega_d) = X^{k+h}_{d,n}(\Omega_d). \tag{9.15}$$

Let us now discuss some Sobolev trace embeddings for concrete spaces which can be derived from the results of this section.
Theorem 9.3 , and its Corollary 9.4, enable us to recover standard trace embeddings for $W^{m,p}(\Omega)$, for every $m \in \mathbb{N}$ and $p \geq 1$, which tell us that

$$\mathrm{Tr} : W^{m,p}(\Omega) \to \begin{cases} L^{\frac{pd}{n-mp}}(\Omega_d) & \text{if } m < n \text{ and } p \in [1, \frac{n}{m}), \\ \exp L^{\frac{n}{n-m}}(\Omega_d) & \text{if } m < n \text{ and } p = \frac{n}{m}, \\ L^\infty(\Omega_d) & \text{otherwise.} \end{cases} \tag{9.16}$$

Equation (9.16) collects classical embedding theorems due to Gagliardo [Ga] ($1 \leq p < \frac{n}{m}$ or $p > \frac{n}{m}$), Nirenberg [Ni] ($p = 1$, $d = n$), Sobolev [So] ($d = n$, and $1 < p < \frac{n}{m}$ or $p > \frac{n}{m}$), Pohozaev [Po], Trudinger [Tr], Yudovich [Yu] ($d = n$, $p = \frac{n}{m}$), Adams [Ad], Maz'ya [Ma2] ($p = \frac{n}{m}$).

More interestingly, a specialization of Theorem 9.3 yields an improvement of the first two trace embeddings in (9.16), which tells us that, if $m < n$, then

$$\text{Tr} : W^{m,p}(\Omega) \to \begin{cases} L^{\frac{pd}{n-mp},p}(\Omega_d) & \text{if } p \in [1, \frac{n}{m}), \\ L^{\infty,\frac{n}{m};-1}(\Omega_d) & \text{if } p = \frac{n}{m}. \end{cases} \tag{9.17}$$

Observe that (9.17) actually strengthens the first two embeddings in (9.16), since $L^{\frac{pd}{n-mp},p}(\Omega_d) \subsetneqq L^{\frac{pd}{n-mp}}(\Omega_d)$ (unless $p = 1$ and $d = n-m$, in which case the two spaces coincide), and $L^{\infty,\frac{n}{m};-1}(\Omega_d) \subsetneqq \exp L^{\frac{n}{n-m}}(\Omega_d)$. Moreover, the target spaces in (9.17) are optimal among all rearrangement-invariant spaces.

The trace embeddings in (9.17) are, in turn, a special instance of the following theorem, dealing with optimal trace embeddings for Lorentz-Sobolev spaces.

Theorem 9.8 [Optimal trace embeddings in Lorentz-Sobolev spaces] [CP2] *Let n, d, m, and Ω be as in Theorem 9.3. Assume that either $p \in (1, \infty)$ and $q \in [1, \infty]$, or $p = q = 1$, or $p = q = \infty$. Then*

$$\text{Tr} : W^m L^{p,q}(\Omega) \to \begin{cases} L^{\frac{pd}{n-mp},q}(\Omega_d) & \text{if } m < n \text{ and } p \in [1, \frac{n}{m}), \\ L^{\infty,q;-1}(\Omega_d) & \text{if } m < n,\ p = \frac{n}{m} \text{ and } q > 1, \\ L^{\infty}(\Omega_d) & \text{otherwise.} \end{cases} \tag{9.18}$$

Moreover, the target spaces in (9.18) *are optimal among all rearrangement-invariant spaces on Ω_d.*

We next focus on trace embeddings for Orlicz-Sobolev spaces. Let $n, m, d \in \mathbb{N}$ be as in the statement of Theorem 9.3. Let A be a Young function, and let $m < n$. We may assume, without loss of generality, that

$$\int_0 \left(\frac{t}{A(t)}\right)^{\frac{m}{n-m}} dt < \infty. \tag{9.19}$$

Indeed, A can be replaced, if necessary, by a Young function equivalent near infinity, which renders (9.19) true, such replacement leaving the Orlicz-Sobolev space $W^m L^A(\Omega)$ unchanged (up to equivalent norms).

If $m < n$, and the integral

$$\int^{\infty} \left(\frac{t}{A(t)}\right)^{\frac{m}{n-m}} dt \tag{9.20}$$

diverges, define the function $H_m : [0, \infty) \to [0, \infty)$ as

$$H_m(s) = \left(\int_0^s \left(\frac{t}{A(t)}\right)^{\frac{m}{n-m}} dt\right)^{\frac{n-m}{n}} \qquad \text{for } s \geq 0, \tag{9.21}$$

and the Young function $A_{m,d}$ by

$$A_{m,d}(t) = \int_0^{H_m^{-1}(t)} \left(\frac{A(s)}{s}\right)^{\frac{d-m}{n-m}} H_m(s)^{\frac{d-n}{n-m}} ds \qquad \text{for } t \geq 0. \tag{9.22}$$

The following result provides us with an optimal Orlicz target in Orlicz-Sobolev trace embeddings. Its proof rests on Theorem 9.1, and on a Hardy type inequality in Orlicz spaces [Ci4, Theorem 3.5].

Theorem 9.9 [Orlicz-Sobolev trace embedding with optimal Orlicz target] [CP2] *Let n, d, m, and Ω be as in Theorem 9.3. Let A be a Young function fulfilling* (9.19). *Then*
(9.23)

$$\mathrm{Tr}: W^m L^A(\Omega) \to \begin{cases} L^{A_{m,d}}(\Omega_d) & \text{if } m<n, \text{ and the integral (9.20) diverges,} \\ L^\infty(\Omega_d) & \text{if either } m \geq n, \text{ or } m<n \text{ and the integral (9.20) converges.} \end{cases}$$

Moreover, the target spaces in (9.23) *are optimal among all Orlicz spaces.*

In the examples below, we present applications of Theorem 9.9 to a couple of customary instances of Orlicz-Sobolev spaces.

Example 9.10 Assume that either $p>1$ and $\alpha \in \mathbb{R}$, or $p=1$ and $\alpha \geq 0$. An application of Theorem 9.9 yields
(9.24)

$$\mathrm{Tr}: W^m L^p(\log L)^\alpha(\Omega) \to \begin{cases} L^{\frac{pd}{n-mp}}(\log L)^{\frac{\alpha d}{n-mp}}(\Omega_d) & \text{if } 1 \leq p < \frac{n}{m} \\ \exp L^{\frac{n}{n-m-\alpha m}}(\Omega_d) & \text{if } p=\frac{n}{m} \text{ and } \alpha < \frac{n-m}{m} \\ \exp\exp L^{\frac{n}{n-m}}(\Omega_d) & \text{if } p=\frac{n}{m} \text{ and } \alpha = \frac{n-m}{m} \\ L^\infty(\Omega_d) & \text{if either } p=\frac{n}{m} \text{ and } \alpha > \frac{n-m}{m}, \text{ or } p > \frac{n}{m}, \end{cases}$$

all the range spaces being optimal in the class of Orlicz spaces.

Example 9.11 Assume that p and α are as in Example 9.10. Then, one can obtain from Theorem 9.9 that

$$\mathrm{Tr}: W^m L^p(\log\,\log L)^\alpha(\Omega) \to \begin{cases} L^{\frac{pd}{n-mp}}(\log\,\log L)^{\frac{\alpha d}{n-mp}} & \text{if } 1 \leq p < \frac{n}{m}, \\ \exp\left(L^{\frac{n}{n-m}}(\log L)^{\frac{\alpha m}{n-m}}\right)(\Omega_d) & \text{if } p = \frac{n}{m}, \\ L^\infty(\Omega_d) & \text{if } p > \frac{n}{m}\,. \end{cases} \tag{9.25}$$

Moreover, the range spaces are sharp in the framework of Orlicz spaces on Ω_d.

We conclude by pointing out that, although the target space in the first embedding in Theorem 9.9 is optimal in the framework of Orlicz spaces, it can be improved if the class of admissible target is enlarged to include all rearrangement-invariant spaces. It turns out that the optimal rearrangement-invariant target space in the first case of (9.23) is an Orlicz-Lorentz space, which can be explicitly exhibited. We refer the reader to [CP2] for this result.

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